

Optimal Adaptive Ridgelet Schemes for Linear Transport Equations

Philipp Grohs, Axel Obermeier
 Seminar for Applied Mathematics, ETH Zürich
`{philipp.grohs,axel.obermeier}@sam.math.ethz.ch`

September 8, 2014

Abstract

In this paper we present a novel method for the numerical solution of linear transport equations, which is based on ridgelets. Such equations arise for instance in radiative transfer or in phase contrast imaging. Due to the fact that ridgelet systems are well adapted to the structure of linear transport operators, it can be shown that our scheme operates in optimal complexity, even if line singularities are present in the solution.

The key to this is showing that the system matrix (with diagonal preconditioning) is uniformly well-conditioned and compressible – the proof for the latter represents the main part of the paper. We conclude with some numerical experiments about N -term approximations and how they are recovered by the solver, as well as localisation of singularities in the ridgelet frame.

Contents

1	Introduction	2
2	Well-Posedness	5
3	Discretisation	8
4	Ridgelet Frames	13
5	Compressibility	16
6	Main results	25
7	Numerical Experiments	25
	Appendices	
A	Geometric Considerations	31
B	A Suitable Choice of Window Functions	37
C	Derivatives and Convolutions	41
	Bibliography	44

1 Introduction

In the past two decades, a wide range of multiscale systems have been introduced with lasting impact in many different fields, starting with wavelets [Dau92] and continuing with ridgelets [Can98], curvelets [CD05b, CD05a, CDDY06], shearlets [KLLW05, KL12], contourlets [DV05] etc. – the latter three of which fall into the framework of so-called “parabolic molecules” [GK14], while all of the mentioned systems are encompassed by the even broader framework of α -molecules [GKKS14].

These systems share the property that they are very well-adapted to representing certain classes of functions optimally (in the sense of the decay rate of the best N -term approximation) – functions with point singularities for wavelets, line singularities for ridgelets and curved singularities for parabolic molecules. Since these classes make up the fundamental phenomenological features of most images in an extremely diverse set of applications, it is perhaps not surprising, that many of the above-mentioned systems were originally investigated in view of their properties regarding image processing.

With a certain time-lag, it is becoming apparent that these systems are also very suitable for solving partial differential equations – again, wavelets were the first in this regard, for example leading to provably optimal solvers for elliptic equations [CDD01]. For differential equations with strong directional features – such as transport equations – it is intuitively clear that optimal solvers will need to take these features into account, however, the development of solvers based on directional systems is still in its infancy.

Following recent results [Gro11], that ridgelets permit the construction of simple diagonal preconditioners for linear transport equations which arise in collocation-type discretization methods for kinetic transport equations (such as radiative transport), we intend this paper (and its companion [EGO14]) to be a first step towards establishing directional representation systems as a useful tool for solving PDEs.

Perhaps the main reason for the success of wavelets in PDE solvers (which, as a long term goal, we would like to emulate) is that they do not only represent typical solutions efficiently, but – crucially – that they simultaneously sparsify (in a suitable sense) the resulting system matrices corresponding to the differential operator and achieve uniformly well-conditioned matrices with simple preconditioning.

The main focus of the present paper is to demonstrate that the same properties hold for ridgelets applied to the numerical discretisation of linear transport equations, and using the machinery of [CDD01] to show that this leads to solvers with optimal complexity.

1.1 Radiative Transport Equation

The motivation for this work is the numerical solution of the following model equation, described by the radiative transport equation (RTE),

$$Au := \vec{s} \cdot \nabla u + \kappa u = f + \int_{\mathbb{S}^{d-1}} \sigma u \, d\vec{s}'. \quad (1.1)$$

It is a steady state continuity equation describing the conservation of radiative intensity in an absorbing, emitting and scattering medium, see e.g. [Mod13]. We will, however, not treat the scattering operator in this paper, which can be incorporated through a variety of methods, not the least of which – the source iteration – we implemented in [EGO14]. Let us assume that the following quantities are known at all locations $\vec{x} \in \Omega \subset \mathbb{R}^d$ and for all directions $\vec{s} \in \mathbb{S}^{d-1} := \{\vec{s} \in \mathbb{R}^d : \|\vec{s}\|_2 = 1\}$:

- absorption coefficient $\kappa(\vec{x}, \vec{s}) \geq \kappa_0 > 0$
- source term $f(\vec{x}, \vec{s}) \in \mathbb{R}$

Then, the above equation allows us to find the unknown radiative intensity u as a function $\Omega \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}$.

Although the RTE looks simple, standard numerical techniques for solving it do not perform well for a number of reasons, mainly:

- The transport term $\vec{s} \cdot \nabla u$ leads to ill-conditioned systems of equations.
- Singularities in the input data may remain in the solution.

- With the dimension of the domain of u being 3 in 2-dimensional physical space and 5 in 3-dimensional space, the problem is fairly high-dimensional.

These issues make the accurate numerical solution of the RTE very costly or even impossible due to memory and compute power limitations of today's hardware.

1.2 Ridgelets

Our proposed approach to solving (1.1), while addressing the above-mentioned problems, is to discretise the equation in physical space using ridgelets. At a glance, a ridgelet is a function which is located along a line, orthogonal to which it oscillates heavily and along which it varies only little (see Figure 1.1a for an example). The idea is to build a basis (or rather, a frame) out of such ridgelets with varying locations, directions and widths, with which we can represent a function whose features are located along curves by a linear combination of relatively few of them. Solutions of the RTE typically fall into this category of functions that can be efficiently represented by such a system, as the variations along the transport direction are smoothed out while the ones orthogonal to it are not – in particular, singularities in the input data may remain.

The present work provides a first step towards a ridgelet-based construction of an optimally convergent numerical solver for (1.1). More precisely we consider the RTE for fixed directions \vec{s} and show that our proposed scheme delivers optimal convergence rates for linear transport equations

$$\vec{s} \cdot \nabla u(\vec{x}) + \kappa(\vec{x})u(\vec{x}) = f(\vec{x}). \quad (1.2)$$

Since a number of numerical methods for the solution of (1.1) heavily relies on efficient solvers of the above linear transport equation, the spatial discretization scheme developed and analyzed in the present paper can be directly utilized for the numerical approximation of solutions to the RTE – as is done in [EGO14].

Before we describe our approach in more detail we would like to pause and comment on its novel properties and limitations.

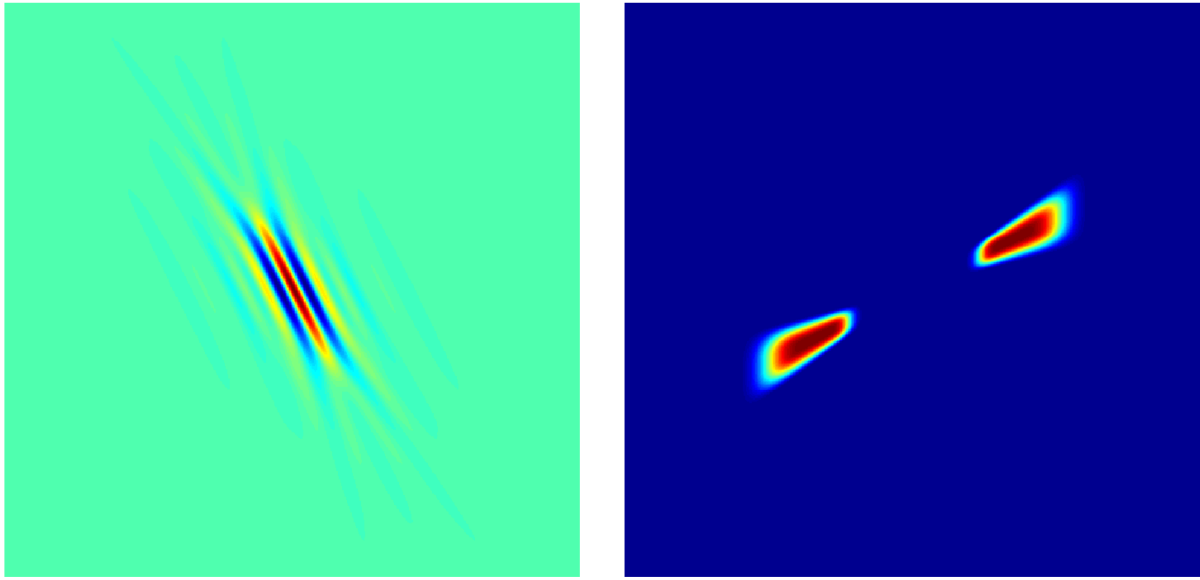
The most important property, and the main result of this paper is the fact that our proposed algorithm is able to approximate solutions u of (1.2) in optimal complexity. In this regard our results are very strong: complexity here is measured in terms of arithmetic operations to be carried out by a processor and the solution is even allowed to possess singularities along lines. Moreover our result hold uniformly in \vec{s} , meaning that they are independent of the transport direction. This property is of essential importance for solving the full RTE.

Even though the PDE (1.2) is of admittedly simple form with several efficient methods to solve it (cf. [EG04]) we are not aware of any method with such strong convergence results as is the case for our proposed scheme. For instance our method converges exponentially for solutions u which are piecewise smooth with a line singularity (see Theorem 6.2) and this result holds uniformly for all directions \vec{s} . Such a result is far from true for conventional (eg. Finite-Element-based) discretization schemes where the expected convergence rate would be of order $N^{-\frac{1}{2}}$ instead, with N being the number of arithmetic operations.

We consider the present paper as a first step in a larger programme of developing ridgelet-based solvers for the RTE. Therefore, in the following paragraphs we outline some limitations of the results as well as some promising directions for future work, opened up by our results.

The convergence results are confined to linear transport equations (1.2) and our analysis assumes that \vec{x} belongs to the full space \mathbb{R}^d . The latter fact poses no problem if for instance the source term f is compactly supported but in many applications one needs to restrict \vec{x} to a finite domain $D \subset \mathbb{R}^d$ and impose inflow boundary conditions. The efficient incorporation of boundary conditions will require the construction of ridgelet frames on finite domains which is the subject of future work (to be more precise, incorporation of inflow boundary conditions is possible with the code developed in [EGO14] but a rigorous analysis is still lacking). With such a construction at hand the theoretical analysis carried out in this paper would essentially go through also for finite domains.

With regard to the fact that the model equation (1.2) addressed in this paper is far simpler than the full radiative transport equation we would like to mention that the paper [EGO14] combines a ridgelet



(a) Physical space (green denotes 0)

(b) Fourier space (blue denotes 0)

Figure 1.1: An illustration of a ridgelet in the two relevant spaces

solver in space with a sparse collocation method to solve the full RTE efficiently. There, a key feature of the use of ridgelets is that collocation in angle leads to uniformly well-conditioned linear systems to be solved, independent of the spatial resolution – a key property for efficient parallelisation. It is possible to go further by combining the spatial ridgelet discretisation as developed in the present paper with a wavelet discretisation on the sphere by a tensor product construction to develop an adaptive numerical algorithm for the full RTE. Again, this is the subject of future work.

1.3 Outline

We begin the paper with a brief investigation of the well-posedness of the main equation in Section 2. In Section 3, we introduce the framework of the discretisation, review how the discretised system can be solved algorithmically, and discuss which properties have to be satisfied to achieve optimal complexity – see Theorem 3.11.

The subsequent Section 4 recalls the ridgelet construction and how it forms a frame for the appropriate spaces, as well as the corresponding preconditioner, leading to the stability result Theorem 4.4.

The core of the paper is in Section 5, where we prove compressibility of the system matrix corresponding to the model problem (1.2) – see Theorem 5.4. Of the necessary properties for optimal complexity mentioned above, this is the key tool to allow approximate linear-time matrix-vector multiplication. Some necessary but less interesting technical details of the proof are outsourced into the appendix.

In the penultimate Section 6, we bring together the separate threads to arrive at the result that – in fact – ridgelets do achieve the desired optimal complexity (Corollary 6.1), and additionally, *also* sparsify typical solutions of such transport equations in the sense of best N -term approximations (Theorem 6.2).

The final Section 7 reports on a proof-of-concept implementation and corresponding numerical experiments.

1.4 Notation

We let $B_X(x, r) := \{x' \in X : \text{dist}_X(x, x') < r\}$ be the open ball in the metric space X . Occasionally we omit the space if it is clear from the context. To distinguish the Euclidian norm from the other norms, we denote it by $|\vec{x}|$. The inner product on \mathbb{R}^d is simply denoted by $\vec{x} \cdot \vec{x}'$, all other inner products are denoted by $\langle \cdot, \cdot \rangle$, where the *first* argument is antilinear and the *second* is linear (which is closer to the interpretation as a functional (see e.g. Bra-ket notation) and has several advantages, in our opinion).

The Fourier transform we use is

$$\hat{f}(\vec{\xi}) := [\mathcal{F}(f)](\vec{\xi}) := \int_{\mathbb{R}^d} f(\vec{x}) e^{-2\pi i \vec{x} \cdot \vec{\xi}} d\vec{x},$$

where we will mostly omit the square brackets for improved legibility if the second term has to be used. In order to limit the amount of constants we have to carry, we define the following relation,

$$A(y) \lesssim B(y) :\iff \exists c > 0 : A(y) \leq cB(y),$$

where the constant has to be independent of y . We try to explicitly state each constant at least once, before swallowing it into the \lesssim -sign. Additionally, $A \sim B$ denotes the case that both $A \lesssim B$ and $B \lesssim A$ hold.

We abbreviate the minimum and maximum of two quantites (if clear from context which two) by $y_< := \min(y, y')$ and $y_> := \max(y, y')$, respectively.

2 Well-Posedness

Starting point is the differential operator

$$A : H^{\vec{s}}(\mathbb{R}^d) \ni u \mapsto \vec{s} \cdot \nabla u(\vec{x}) + \kappa(\vec{x})u(\vec{x}) \in L^2(\mathbb{R}^d)$$

with fixed $\vec{s} \in \mathbb{S}^{d-1}$ and a function $\kappa \in L^\infty(\mathbb{R}^d)$ that satisfies $\kappa(\vec{x}) \geq \gamma > 0$, $\forall \vec{x} \in \mathbb{R}^d$. The space $H^{\vec{s}}$ is defined as follows.

Definition 2.1. Let $\vec{s} \in \mathbb{S}^{d-1}$, then we define the *anisotropic Sobolev space*

$$H^{k+\vec{s}}(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) : (\vec{s} \cdot \nabla)f \in H^k(\mathbb{R}^d)\},$$

where $H^k(\mathbb{R}^d)$ is the usual Sobolev space. It is equipped with the norm

$$\|f\|_{H^{k+\vec{s}}(\mathbb{R}^d)}^2 := \|f\|_{H^k(\mathbb{R}^d)}^2 + \|(\vec{s} \cdot \nabla)f\|_{H^k(\mathbb{R}^d)}^2.$$

We set $H^{\vec{s}} := H^{0+\vec{s}}$. These spaces are more easily characterised on the Fourier side,

$$H^{k+\vec{s}}(\mathbb{R}^d) := \left\{ \hat{f} \in L^2(\mathbb{R}^d) : \langle \vec{s} \cdot \vec{\xi} \rangle \langle \vec{\xi} \rangle^k \hat{f}(\hat{x}, \hat{y}) \in L^2(\mathbb{R}^d) \right\}$$

with norm

$$\|\hat{f}\|_{H^{k+\vec{s}}(\mathbb{R}^d)} := \|\langle \vec{s} \cdot \vec{\xi} \rangle \langle \vec{\xi} \rangle^k \hat{f}\|_{L^2(\mathbb{R}^d)}.$$

To make the operators involved positive definite, we have to restrict ourselves to solving the normal equation $A^*Au = A^*f \in L^2(\mathbb{R}^d)$, which we do by minimising the L^2 -residual,

$$u_0 = \underset{v \in H^{\vec{s}}}{\operatorname{argmin}} \|A^*Av - A^*f\|_{L^2}. \quad (2.1)$$

Theorem 2.2. *The problem of finding $u \in H^{\vec{s}}$ such that $Au = f \in L^2(\mathbb{R}^d)$ is well-posed. In addition, for $u \in H^{\vec{s}}$, the following norm-equivalence holds*

$$\|Au\|_{L^2} \sim \|u\|_{H^{\vec{s}}}. \quad (2.2)$$

Before we come to the proof of Theorem 2.2 we introduce some notation. Let $R_{\vec{s}}$ be an orthogonal matrix which maps \vec{s} to $\vec{e}_1 = (1, 0, \dots)^\top$, and let $R_{\vec{s}}^{-1} = R_{\vec{s}}^\top$ be its inverse. This rotation is not unique for $d > 3$ (see also Remark 4.6), however, an arbitrary but fixed choice suffices for this section. We define the respective pullbacks for $f \in L^2(\mathbb{R}^d)$ by

$$\rho_{\vec{s}}f(\vec{x}) := f(R_{\vec{s}}^{-1}\vec{x}), \quad \rho_{\vec{s}}^{-1}f(\vec{x}) := f(R_{\vec{s}}\vec{x}),$$

thus $\rho_{\vec{s}}f(\vec{e}_1) = f(\vec{s})$, $\rho_{\vec{s}}^{-1}f(\vec{s}) = f(\vec{e}_1)$ – if f is continuous. These pullbacks is also well-defined for $L^2(\mathbb{R}^d)$ -functions, as long as we don't evaluate at a single value – since we always integrate in the following, this presents no problem.

We will use these transformations to restrict ourselves to dealing with just the derivative in the first component x_1 , as the following lemma shows.

Lemma 2.3. *For $u \in H^{\vec{s}}(\mathbb{R}^d)$,*

$$\vec{s} \cdot \nabla u = \rho_{\vec{s}}^{-1} \frac{d}{dx_1} (\rho_{\vec{s}}u)(\vec{x}). \quad (2.3)$$

Proof. Our notation for the Jacobian is

$$dg(\vec{x}) = \left(\frac{\partial g_i}{\partial x_j}(\vec{x}) \right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \quad \text{for } g : \mathbb{R}^n \rightarrow \mathbb{R}^m,$$

whereby $\frac{d}{d\vec{s}}g(\vec{x}) = (dg(\vec{x}))\vec{s}$, and the chain rule is written as $d(g \circ h)(\vec{x}) = dg(h(\vec{x}))dh(\vec{x})$ for $h : \mathbb{R}^\ell \rightarrow \mathbb{R}^n$. If $m = 1$, the vector is usually written upright, of course, i.e. $\nabla g = (dg(\vec{x}))^\top$. Thus,

$$\frac{d}{dx_1}(\rho_{\vec{s}}u)(\vec{x}) = (d(u \circ R_{\vec{s}}^{-1})(\vec{x}))\vec{e}_1 = du(R_{\vec{s}}^{-1}\vec{x}) \underbrace{R_{\vec{s}}^{-1}\vec{e}_1}_{=\vec{s}} = \vec{s} \cdot \nabla u(R_{\vec{s}}^{-1}\vec{x}) = \rho_{\vec{s}}(\vec{s} \cdot \nabla u)(\vec{x}).$$

Applying $\rho_{\vec{s}}^{-1}$ yields the result. \square

Remark 2.4. An immediate consequence of Lemma 2.3 is

$$u \in H^{\vec{s}}(\mathbb{R}^d) \iff \rho_{\vec{s}}u \in H^{\vec{e}_1}(\mathbb{R}^d).$$

Proof of Theorem 2.2. This proof is a simple adaptation of the proof in [GS11]. By the previous lemma we immediately see that

$$Au = f \iff \rho_{\vec{s}}Au = \rho_{\vec{s}}f \iff \frac{d}{dx_1}(\rho_{\vec{s}}u) + \rho_{\vec{s}}\kappa \rho_{\vec{s}}u = \rho_{\vec{s}}f. \quad (2.4)$$

Using variation of constants, this can be solved explicitly for arbitrary $f \in L^2(\mathbb{R}^d)$ in the following way. We let $\vec{x}' := (x_2, \dots, x_d)^\top \in \mathbb{R}^{d-1}$ be the vector of the $d - 1$ lower components of $\vec{x} = \begin{pmatrix} x_1 \\ \vec{x}' \end{pmatrix}$ and compute

$$y(x_1, \vec{x}') := e^{-K(x_1, \vec{x}')} \left(\int_0^{x_1} \rho_{\vec{s}}f(t, \vec{x}') e^{K(t, \vec{x}')} dt + C \right), \quad \text{where} \quad K(t, \vec{x}') = \int_0^t \rho_{\vec{s}}\kappa(r, \vec{x}') dr.$$

Note that since $\kappa \geq \gamma > 0$, K is a strictly increasing function of x_1 (with slope at least γ), in particular $K(t, \vec{x}') - K(x_1, \vec{x}') \leq \gamma(t - x_1)$ for $t \leq x_1$.

In general, y will not be in $L^2(\mathbb{R}^d)$ – something we clearly need. As a necessary requirement, it must tend to zero for large negative x_1 , which – considering the exponential growth of the first factor – means that the second factor must tend to zero, thus determining the constant C ;

$$y(x_1, \vec{x}') \xrightarrow{x_1 \rightarrow -\infty} 0, \implies C = \int_{-\infty}^0 \rho_{\vec{s}} f(t, \vec{x}') e^{K(t, \vec{x}')} dt.$$

For arbitrary $g \in L^2(\mathbb{R}^d)$ we compute

$$\begin{aligned} |\langle y, g \rangle_{L^2}| &= \left| \int_{\mathbb{R}^d} g(\vec{x}) \int_{-\infty}^{x_1} \rho_{\vec{s}} f(t, \vec{x}') e^{K(t, \vec{x}') - K(x_1, \vec{x}')} dt d\vec{x} \right| \leq \int_{\mathbb{R}^d} \int_{-\infty}^{x_1} |g(\vec{x})| |\rho_{\vec{s}} f(t, \vec{x}')| e^{K(t, \vec{x}') - K(x_1, \vec{x}')} dt d\vec{x} \\ &\leq \int_{\mathbb{R}^d} \int_{-\infty}^{x_1} |g(\vec{x})| |\rho_{\vec{s}} f(t, \vec{x}')| e^{\gamma(t-x_1)} dt d\vec{x} = \int_{\mathbb{R}^d} \int_{-\infty}^0 |g(\vec{x})| |\rho_{\vec{s}} f(x_1 + r, \vec{x}')| e^{\gamma r} dr d\vec{x} \\ &\stackrel{(*)}{=} \int_{-\infty}^0 \int_{\mathbb{R}^d} |g(\vec{x})| |\rho_{\vec{s}} f(x_1 + r, \vec{x}')| e^{\gamma r} d\vec{x} dr \stackrel{(**)}{\leq} \|g\|_{L^2} \|\rho_{\vec{s}} f\|_{L^2} \int_{-\infty}^0 e^{\gamma r} dr = \frac{1}{\gamma} \|g\|_{L^2} \|f\|_{L^2}, \end{aligned}$$

where for $(*)$ we used Fubini's theorem, while $(**)$ makes use of the Cauchy–Schwarz inequality, as well as the fact that translating the argument of a function $f \in L^2(\mathbb{R}^d)$ by a fixed vector (in this case $(r, 0, \dots)^\top$) preserves the norm. By the Riesz representation theorem, we see $\|y\|_{L^2} \leq \frac{1}{\gamma} \|f\|_{L^2}$, and in particular that $y \in H^{\vec{s}}$, since differentiability is obvious from the construction. Setting $u := \rho_{\vec{s}}^{-1} y$, (2.4) shows that we have found a solution of $Au = f$ for arbitrary $f \in L^2(\mathbb{R}^d)$, which is what we wanted to prove.

To see (2.2), we consider the norm of the derivative

$$\left\| \frac{d}{dx_1} \rho_{\vec{s}} u \right\|_{L^2} = \|\rho_{\vec{s}} f - \rho_{\vec{s}} \kappa \rho_{\vec{s}} u\|_{L^2} \leq \|f\|_{L^2} + \|\kappa\|_{L^\infty} \|u\|_{L^2} \lesssim \|f\|_{L^2},$$

where the last inequality is due to $\|u\|_{L^2} = \|y\|_{L^2} \leq \frac{1}{\gamma} \|f\|_{L^2}$. Consequently,

$$\|\rho_{\vec{s}} u\|_{H^{\vec{s}_1}}^2 = \|\rho_{\vec{s}} u\|_{L^2}^2 + \left\| \frac{d}{dx_1} \rho_{\vec{s}} u \right\|_{L^2}^2 \lesssim \|f\|_{L^2}^2 = \|\rho_{\vec{s}} f\|_{L^2}^2 \stackrel{(2.4)}{=} \int_{\mathbb{R}^d} \left| \frac{d}{dx_1} \rho_{\vec{s}} u(\vec{x}) + \rho_{\vec{s}} \kappa(\vec{x}) \rho_{\vec{s}} u(\vec{x}) \right|^2 d\vec{x}, \quad (2.5)$$

and putting everything together (as well as substituting twice), we arrive at

$$\begin{aligned} \|u\|_{H^{\vec{s}}}^2 &= \int_{\mathbb{R}^d} |\vec{s} \cdot \nabla u(\vec{x})|^2 + |u(\vec{x})|^2 d\vec{x} \stackrel{(2.3)}{=} \int_{\mathbb{R}^d} \left| \rho_{\vec{s}}^{-1} \frac{d}{dx_1} (\rho_{\vec{s}} u)(\vec{x}) \right|^2 + |u(\vec{x})|^2 d\vec{x} \\ &= \int_{\mathbb{R}^d} \left(\left| \frac{d}{dx_1} (\rho_{\vec{s}} u)(\vec{x}) \right|^2 + |\rho_{\vec{s}} u(\vec{x})|^2 \right) \underbrace{|\det R_{\vec{s}}^{-1}|}_{=1} d\vec{x} \stackrel{(2.5)}{\lesssim} \int_{\mathbb{R}^d} \left| \frac{d}{dx_1} (\rho_{\vec{s}} u)(\vec{x}) + \rho_{\vec{s}} \kappa(\vec{x}) \rho_{\vec{s}} u(\vec{x}) \right|^2 d\vec{x} \\ &= \int_{\mathbb{R}^d} \left| \rho_{\vec{s}}^{-1} \frac{d}{dx_1} (\rho_{\vec{s}} u)(\vec{x}) + \kappa(\vec{x}) u(\vec{x}) \right|^2 |\det R_{\vec{s}}| d\vec{x} = \int_{\mathbb{R}^d} |\vec{s} \cdot \nabla u(\vec{x}) + \kappa(\vec{x}) u(\vec{x})|^2 d\vec{x} = \|Au\|_{L^2}^2. \end{aligned}$$

The second inequality necessary for (2.2) is immediate,

$$\|Au\|_{L^2} = \|\vec{s} \cdot \nabla u + \kappa u\|_{L^2} \leq \|\vec{s} \cdot \nabla u\|_{L^2} + \|\kappa\|_{L^\infty} \|u\|_{L^2} \lesssim \|u\|_{H^{\vec{s}}},$$

and thus we have shown the equivalence of the norms, $\|Au\|_{L^2} \sim \|u\|_{H^{\vec{s}}}$, which finishes the proof. \square

Corollary 2.5. *For every $\ell \in (H^{\vec{s}})'$ – the dual of $H^{\vec{s}}$ – there exists a unique $u_0 \in H^{\vec{s}}$ which solves (2.1). Moreover, the solution is characterized by the variational equation*

$$a(v, u_0) = \ell(v) \quad \text{for all } v \in H^{\vec{s}}, \quad (2.6)$$

where we put

$$a(v, u) := \langle Av, Au \rangle_{L^2}.$$

In particular, well-definedness holds for

$$\ell_f(v) := \langle Av, f \rangle_{L^2} \quad \text{with } f \in L^2(\mathbb{R}^d). \quad (2.7)$$

Proof. The first statement is a direct consequence of Theorem 2.2 (yielding continuity and coercivity of a in terms of $\|\cdot\|_{H^{\vec{s}}}$) and the Lax-Milgram lemma. Equation (2.6) is simply a reformulation as a linear least squares problem. Finally, well-definedness for (2.7) holds, since ℓ as defined in (2.7) is trivially continuous, as can be seen from the Cauchy-Schwarz inequality. \square

Corollary 2.5 shows that, using L^2 -regularization, we may interpret the operator A^*A as a bounded and boundedly invertible operator $A^*A : H^{\vec{s}} \rightarrow (H^{\vec{s}})'$.

3 Discretisation

In our paper we aim to solve (2.1) via solving a discretization of the linear system (2.6). Several ingredients are needed to render this approach efficient:

- (i) Uniform well-conditionedness of the resulting infinite discrete linear system
- (ii) Fast approximate matrix-vector multiplication for the discrete operator matrix
- (iii) Efficient approximation of typical solutions

There exists several results which essentially state that, whenever (i), (ii) and (iii) are satisfied, then the linear system (2.6) can be solved in optimal computational complexity [CDD01, Ste04, DFR07]. We will formalize what is precisely meant by properties (i)–(iii) later on, but first we need to introduce some further notation.

3.1 Gelfand Frames

Following [DFR07], we will use the concept of a *Gelfand frame* to discretize (2.6). Our starting point is a bounded and boundedly invertible operator

$$F : \mathcal{H} \rightarrow \mathcal{H}', \quad (3.1)$$

for some Hilbert space \mathcal{H} , inducing a symmetric and coercive bilinear form,

$$a(u, v) = \langle Fu, v \rangle_{\mathcal{H}' \times \mathcal{H}}, \quad a(v, v) \sim \|v\|_{\mathcal{H}}^2,$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}' \times \mathcal{H}}$ is the duality pairing of \mathcal{H}' and \mathcal{H} . The aim is to provide an efficient discretization of this operator.

To do this we first consider discrete systems $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda}$ which provide a stable decomposition and reconstruction procedure, so-called *frames*:

Definition 3.1. Let Λ be a discrete set and \mathcal{H} a Hilbert space. A system $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda}$ with $\varphi_\lambda \in \mathcal{H}$ for all $\lambda \in \Lambda$ is called a *frame* if there exist constants $0 < c_\Phi \leq C_\Phi < \infty$ such that

$$c_\Phi \|f\|_{\mathcal{H}}^2 \leq \sum_{\lambda \in \Lambda} |\langle \varphi_\lambda, f \rangle_{\mathcal{H}}|^2 \leq C_\Phi \|f\|_{\mathcal{H}}^2.$$

If $c_\Phi = C_\Phi$ one calls Φ a *tight frame*; if, additionally $c_\Phi = 1$, one speaks of a *Parseval frame*.

For a frame Φ for \mathcal{H} we also need to define the *frame analysis operator*

$$G : \begin{cases} \mathcal{H} & \rightarrow \ell^2(\Lambda) \\ f & \mapsto \langle \Phi, f \rangle_{\mathcal{H}} := (\langle \varphi_\lambda, f \rangle_{\mathcal{H}})_{\lambda \in \Lambda}, \end{cases}$$

and its dual the *frame reconstruction operator*

$$G^* : \begin{cases} \ell^2(\Lambda) & \rightarrow \mathcal{H} \\ \mathbf{c} & \mapsto \Phi \mathbf{c} := \sum_{\lambda \in \Lambda} c_\lambda \varphi_\lambda. \end{cases}$$

The definition of a frame implies that the operator

$$S_\Phi : \mathcal{H} \rightarrow \mathcal{H}, \quad f \mapsto G^* G f$$

is symmetric, bounded and boundedly invertible. The *canonical dual frame* of Φ is defined as $\tilde{\Phi} := S_\Phi^{-1} \Phi \subseteq \mathcal{H}$.

Additionally, we need the notion of a *Gelfand triple*:

Definition 3.2. Let \mathcal{H} be a Hilbert space with dual \mathcal{H}' . If we have

$$\mathcal{H} \subseteq L^2(\Omega) \subseteq \mathcal{H}'$$

with \mathcal{H} a Hilbert space such that all inclusions above are continuous and dense, then the triplet $(\mathcal{H}, L^2(\Omega), \mathcal{H}')$ is called a *Gelfand triple*.

Remark 3.3. A canonical example for a Gelfand triple is induced by the Sobolev space $\mathcal{H} = H_0^1(\Omega)$ for some domain $\Omega \subseteq \mathbb{R}^d$. Of more interest to our purpose is the case

$$\mathcal{H} = H^{\vec{s}}(\mathbb{R}^d),$$

which also induces a Gelfand triple.

The concept of Gelfand triples is actually much more general and allows for $\mathcal{B} \subseteq \mathcal{H} \subseteq \mathcal{B}'$ where \mathcal{B} is a Banach space and \mathcal{H} a Hilbert space (with the same requirement on the embeddings). However, since we need a Hilbert space for the results of [DFR07] in relation to (3.1) anyway, we omit this.

We call a frame $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda}$ for \mathcal{H} a *Gelfand frame* if $\Phi \subseteq \mathcal{H}$, and there exists a Gelfand triple $(\mathcal{H}_d, \ell^2(\Lambda), \mathcal{H}'_d)$ of sequence spaces such that the operators

$$G_\Phi^* : \begin{cases} \mathcal{H}_d & \rightarrow \mathcal{H} \\ \mathbf{c} & \mapsto \Phi \mathbf{c} \end{cases} \quad \text{and} \quad G_{\tilde{\Phi}} : \begin{cases} \mathcal{H} & \rightarrow \mathcal{H}_d \\ f & \mapsto \langle \tilde{\Phi}, f \rangle_{\mathcal{H}' \times \mathcal{H}} = \langle \tilde{\Phi}, f \rangle_{L^2} \end{cases}$$

are bounded. By duality, the operators

$$G_\Phi : \begin{cases} \mathcal{H}' & \rightarrow \mathcal{H}'_d \\ f & \mapsto \langle \Phi, f \rangle_{\mathcal{H}' \times \mathcal{H}} \end{cases} \quad \text{and} \quad G_{\tilde{\Phi}}^* : \begin{cases} \mathcal{H}'_d & \rightarrow \mathcal{H}' \\ \mathbf{c} & \mapsto \tilde{\Phi} \mathbf{c} \end{cases}$$

are also bounded. In addition, suppose that there exists an isomorphism $D_{\mathcal{H}} : \mathcal{H}_d \rightarrow \ell^2(\Lambda)$ such that its $\ell^2(\Lambda)$ -adjoint $D_{\mathcal{H}}^* : \ell^2(\Lambda) \rightarrow \mathcal{H}'_d$ is also an isomorphism.

Now assume that we want to solve the operator equation

$$F u = f \tag{3.2}$$

where $f \in \mathcal{H}'$ and F given above in (3.1). Using a Gelfand frame Φ we can discretize (3.2) to yield the discrete system

$$\mathbf{F} \mathbf{u} = \mathbf{f}, \tag{3.3}$$

with

$$\mathbf{F} = (D_{\mathcal{H}}^*)^{-1} G_\Phi F G_\Phi^* D_{\mathcal{H}}^{-1} \quad \text{and} \quad \mathbf{f} = (D_{\mathcal{H}}^*)^{-1} G_\Phi f.$$

We have the following result which states that the discrete version (3.3) yields a uniformly well-conditioned infinite linear system.

Lemma 3.4 ([DFR07, Lemma 4.1]). *The operator $\mathbf{F} : \ell^2(\Lambda) \rightarrow \ell^2(\Lambda)$ is bounded and boundedly invertible on its range $\text{ran}(\mathbf{F}) = \text{ran}((D_{\mathcal{H}}^*)^{-1} G_\Phi)$. Furthermore, $\ker(\mathbf{F}) = \ker(G_\Phi^* D_{\mathcal{H}}^{-1})$.*

3.2 Numerical Solution of the Discrete System

In the previous subsection we have reformulated the operator equation (3.2) in terms of a discrete linear system

$$\mathbf{F}\mathbf{u} = \mathbf{f}, \quad (3.4)$$

with \mathbf{F} and \mathbf{f} given as above (in particular, this means that $\mathbf{f} \in \text{ran}(\mathbf{F})$ and that \mathbf{F} is positive definite). If we were able to compute with infinite vectors, at this point we could simply use a standard iterative solver such as a damped Richardson iteration

$$\mathbf{u}^{(j+1)} = \mathbf{u}^{(j)} - \alpha(\mathbf{F}\mathbf{u}^{(j)} - \mathbf{f}), \quad \mathbf{u}^{(0)} = \mathbf{0}. \quad (3.5)$$

Due to the well-conditionedness of the matrix \mathbf{F} ensured by Lemma 3.4 and the fact that the iterates stay in $\text{ran}(\mathbf{F})$ in each step, it is easy to show that for appropriate damping α the sequence $\mathbf{u}^{(j)}$ converges geometrically to the sought solution \mathbf{u} in the $\ell^2(\Lambda)$ -norm, i.e.

$$\|\mathbf{u} - \mathbf{u}^{(j)}\|_{\ell^2(\Lambda)} \lesssim \rho^j$$

for some $\rho < 1$, depending on the spectral properties of the operator \mathbf{F} .

In view of a practical realization of the above scheme, two fundamental issues arise:

- (A) We only have finite computing capabilities at our disposal, and therefore all operations in (3.5) can only be carried out approximatively
- (B) Due to the approximate computation of the iteration (3.5), we might fall out of $\text{ran}(\mathbf{F})$ during iteration. A consequence is that an error in $\ker(\mathbf{F})$ might not be reduced in subsequent iterations.

In the remainder of this section we discuss how these two issues can be dealt with, without compromising numerical accuracy. We start with (A) which is by now classical for wavelet discretizations of elliptic PDEs. The approximative evaluation of the Richardson iteration utilizes the following three procedures:

- **RHS** $[\varepsilon, \mathbf{f}] \rightarrow \mathbf{f}_\varepsilon$: determines for $\mathbf{f} \in \ell^2(\Lambda)$ a finitely supported $\mathbf{f}_\varepsilon \in \ell^2(\Lambda)$ such that

$$\|\mathbf{f} - \mathbf{f}_\varepsilon\|_{\ell^2(\Lambda)} \leq \varepsilon;$$

- **APPLY** $[\varepsilon, \mathbf{A}, \mathbf{v}] \rightarrow \mathbf{v}_\varepsilon$: determines for $\mathbf{A} : \ell^2(\Lambda) \rightarrow \ell^2(\Lambda)$ and for a finitely supported $\mathbf{v} \in \ell^2(\Lambda)$ a finitely supported \mathbf{v}_ε such that

$$\|\mathbf{A}\mathbf{v} - \mathbf{v}_\varepsilon\|_{\ell^2(\Lambda)} \leq \varepsilon;$$

- **COARSE** $[\varepsilon, \mathbf{c}] \rightarrow \mathbf{c}_\varepsilon$: determines for a finitely supported $\mathbf{u} \in \ell^2(\Lambda)$ a finitely supported $\mathbf{u}_\varepsilon \in \ell^2(\Lambda)$ with at most N nonzero coefficients (by setting the other entries to zero), such that

$$\|\mathbf{c} - \mathbf{c}_\varepsilon\|_{\ell^2(\Lambda)} \leq \varepsilon. \quad (3.6)$$

Moreover, if N_{\min} is the minimal number of coefficients necessary to achieve (3.6), the output achieves $N \lesssim N_{\min}$ in linear time (whereas satisfying N_{\min} would incur an additional log-factor in the complexity).

We refer to [CDD01, Ste04, DFR07] for information on the numerical realization of these routines. Assuming the existence of numerical procedures as above, we can formulate the first numerical algorithm to solve the

discrete linear system (3.4) up to accuracy $\varepsilon > 0$, given as Algorithm 3.5 below.

Algorithm 3.5: Inexact Damped Richardson Iteration

Data: $\varepsilon > 0, \mathbf{F}, \mathbf{f}$

Result: $\mathbf{u}_\varepsilon = \text{SOLVE}[\varepsilon, \mathbf{F}, \mathbf{f}]$

Let $\theta < \frac{1}{3}$ and $K \in \mathbb{N}$ such that $3\rho^K < \theta$. $i := 0, \mathbf{u}^{(0)} := \mathbf{0}, \varepsilon_0 := \|\mathbf{F}|_{\text{ran}(\mathbf{F})}^{-1}\| \|\mathbf{f}\|_{\ell^2(\Lambda)}$

while $\varepsilon_i > \varepsilon$ **do**

$i := i + 1;$

$\varepsilon_i := 3\rho^K \varepsilon_{i-1} / \theta;$

$\mathbf{f}^{(i)} := \text{RHS}[\theta \varepsilon_i / (6\alpha K), \mathbf{f}];$

$\mathbf{u}^{(i,0)} := \mathbf{u}^{(i-1)};$

for $j = 1, \dots, K$ **do**

$\mathbf{u}^{(i,j)} := \mathbf{u}^{(i,j-1)} - \alpha(\text{APPLY}[\theta \varepsilon_i / (6\alpha K), \mathbf{F}, \mathbf{u}^{(i,j-1)}] - \mathbf{f}^{(i)});$

$\mathbf{u}^{(i)} := \text{COARSE}[(1 - \theta)\varepsilon_i, \mathbf{u}^{(i,K)}];$

$\mathbf{u}_\varepsilon := \mathbf{u}^{(i)};$

Conditional on the three routines above, we have thus formulated a feasible algorithm for the approximate solution of (3.4). We will talk about the computational complexity and accuracy of this algorithm in a moment, but first let us discuss the issue (B), namely that errors in $\ker(\mathbf{F})$ may not be decreased during the iterations in Algorithm 3.5. In [Ste04] this problem is addressed and in particular it is shown that possibly the computational complexity of Algorithm 3.5 may deteriorate unless some additional conditions are satisfied. While it is believed that those conditions – most notably the compressibility of the orthogonal projection – are valid, it is impossible to prove them at this time.

3.2.1 The modSOLVE-Algorithm

A remedy is to apply a bounded projection \mathbf{P} such that

$$\ker(\mathbf{P}) = \ker(\mathbf{F}) \stackrel{\text{Lemma 3.4}}{=} \ker(G_\Phi^* D_\mathcal{H}^{-1}) \quad (3.7)$$

every few Richardson iterations in order to remove unwanted error components in $\ker(\mathbf{F})$.

The following discussion also applies to general Gelfand triples $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$, however, we continue in the notation so far (requiring Hilbert instead of Banach spaces), again using the Gelfand frame Φ with canonical dual $\tilde{\Phi}$.

In order to arrive at a projector satisfying (3.7) we consider the (injective) mapping

$$Z : \begin{cases} \mathcal{B} & \rightarrow \ell^2(\Lambda) \\ f & \mapsto D_\mathcal{H} G_{\tilde{\Phi}} f \end{cases}$$

By the definition of a Gelfand frame, this mapping is bounded. We also have that

$$G_\Phi^* D_\mathcal{H}^{-1} Z f = G_\Phi^* D_\mathcal{H}^{-1} D_\mathcal{H} G_{\tilde{\Phi}} f = G_\Phi^* G_{\tilde{\Phi}} f = f \quad \text{for all } f \in \mathcal{H}. \quad (3.8)$$

Therefore, we can put

$$\mathbf{P} := \begin{cases} \ell^2(\Lambda) & \rightarrow \ell^2(\Lambda) \\ \mathbf{c} & \mapsto Z G_\Phi^* D_\mathcal{H}^{-1} \mathbf{c} \end{cases}$$

and see, using (3.8), that this mapping is indeed a projector with

$$\ker(\mathbf{P}) = \ker(G_\Phi^* D_\mathcal{H}^{-1}),$$

which is exactly what we wanted.

To find the matrix representation of \mathbf{P} we note that

$$\mathbf{P} = D_{\mathcal{H}} \langle \tilde{\Phi}, \Phi \rangle_{L^2} D_{\mathcal{H}}^{-1}.$$

Using the projection operator \mathbf{P} as just defined, we now follow [Ste04] and formulate a slightly modified algorithm to approximatively solve (3.4) in Algorithm 3.6.

Algorithm 3.6: Modified Inexact Damped Richardson Iteration

Data: $\varepsilon > 0, \mathbf{F}, \mathbf{f}$

Result: $\mathbf{u}_\varepsilon = \text{modSOLVE}[\varepsilon, \mathbf{F}, \mathbf{P}, \mathbf{f}]$

Let $\theta < \frac{1}{3}$ and $K \in \mathbb{N}$ such that $3\rho^K \|\mathbf{P}\| < \theta$. $i := 0, \mathbf{u}^{(0)} := \mathbf{0}, \varepsilon_0 := \|\mathbf{P}\| \|\mathbf{F}|_{\text{ran}(\mathbf{F})}^{-1}\| \|\mathbf{f}\|_{\ell^2(\Lambda)}$

while $\varepsilon_i > \varepsilon$ **do**

$i := i + 1;$

$\varepsilon_i := 3\rho^K \|\mathbf{P}\| \varepsilon_{i-1} / \theta;$

$\mathbf{f}^{(i)} := \text{RHS}[\theta \varepsilon_i / (6\alpha K \|\mathbf{P}\|), \mathbf{f}];$

$\mathbf{u}^{(i,0)} := \mathbf{u}^{(i-1)};$

for $j = 1, \dots, K$ **do**

$\mathbf{u}^{(i,j)} := \mathbf{u}^{(i,j-1)} - \alpha(\text{APPLY}[\theta \varepsilon_i / (6\alpha K \|\mathbf{P}\|), \mathbf{F}, \mathbf{u}^{(i,j-1)}] - \mathbf{f}^{(i)});$

$\mathbf{z}^{(i)} := \text{APPLY}[\theta \varepsilon_i / 3, \mathbf{P}, \mathbf{u}^{(i,K)}];$

$\mathbf{u}^{(i)} := \text{COARSE}[(1 - \theta) \varepsilon_i, \mathbf{z}^{(i)}];$

$\mathbf{u}_\varepsilon := \mathbf{u}^{(i)};$

3.2.2 Complexity Analysis

We now turn to a complexity analysis of the algorithms **SOLVE** and **modSOLVE** introduced above. To this end it is convenient to work with so-called *weak ℓ^p -spaces*.

Definition 3.7. For $0 < p < 2$ we define the weak ℓ^p -space – denoted by $\ell_w^p(\Lambda)$ – as

$$\ell_w^p(\Lambda) := \{\mathbf{c} \in \ell^2(\Lambda) : |\mathbf{c}|_{\ell_w^p(\Lambda)} := \sup_{n \in \mathbb{N}} n^{\frac{1}{p}} |\gamma_n(\mathbf{c})| < \infty\},$$

where $\gamma_n(\mathbf{c})$ denotes the n -th largest coefficient in modulus of \mathbf{c} .

Remark 3.8. The quasi-Banach spaces ℓ_w^p are instrumental in the study of nonlinear best N -term approximation. More precisely, membership of the coefficient sequence in ℓ_w^p is equivalent to a best N -term approximation rate of order $N^{-\sigma}$, where $\sigma = \frac{1}{p} - \frac{1}{2}$, see [DeV98]. Moreover, it is easy to see that we have the inclusions

$$\ell^p \subseteq \ell_w^p \subseteq \ell^{p+\varepsilon}$$

for any $\varepsilon > 0$.

To achieve optimal convergence rates for our problem through the techniques introduced in [CDD01], a key ingredient is *compressibility* of the discretized operator equation. Such a property guarantees the existence of linear-time approximate matrix-vector multiplication algorithms **APPLY** which are used in the iterative solution of the operator equation, see [CDD01, Ste04] for more information.

Definition 3.9. A matrix \mathbf{A} is called σ^* -compressible if for every $\sigma < \sigma^*$ and $k \in \mathbb{N}$ there exists a matrix $\mathbf{A}^{[k]}$ such that

- (i) the matrix $\mathbf{A}^{[k]}$ has at most $\alpha_k 2^k$ non-zero entries in each column,

(ii) we have

$$\|\mathbf{A} - \mathbf{A}^{[k]}\|_2 \leq C_k$$

so that the sequences $(\alpha_k)_{k \in \mathbb{N}}$, $(C_k 2^{\sigma k})_{k \in \mathbb{N}}$ are both summable.

Definition 3.10 ([Ste04, Def. 3.9]). A vector $\mathbf{c} \in \ell^2$ is called σ^* -optimal, when for a suitable routine **RHS**, for each $\sigma \in (0, \sigma^*)$ with $p := (\frac{1}{2} + \sigma)^{-1}$, the following is valid for $\mathbf{c}_\varepsilon = \mathbf{RHS}[\varepsilon, \mathbf{c}]$:

1. $\#\text{supp } \mathbf{c}_\varepsilon \lesssim \varepsilon^{-1/\sigma} |\mathbf{c}|_{\ell_w^p}^{1/\sigma}$
2. The number of arithmetic operations used to compute \mathbf{c}_ε is at most a multiple of $\varepsilon^{-1/\sigma} |\mathbf{c}|_{\ell_w^p}^{1/\sigma}$.

We can now formulate the main result of [Ste04, Theorem 3.11].

Theorem 3.11 (Convergence of **modSOLVE**). Assume that for some $\sigma^* > 0$, the matrices \mathbf{F} and \mathbf{P} are σ^* -compressible and that for some $\sigma \in (0, \sigma^*)$ and $p := \frac{1}{\frac{1}{2} + \sigma}$, the system $\mathbf{F}\mathbf{u} = \mathbf{f}$ has a solution $\mathbf{u} \in \ell_w^p(\Lambda)$. Moreover, assume that \mathbf{f} is σ^* -optimal. Then for all $\varepsilon > 0$, $\mathbf{u}_\varepsilon := \mathbf{modSOLVE}[\varepsilon, \mathbf{F}, \mathbf{P}, \mathbf{f}]$ satisfies

$$(I) \quad \#\text{supp } \mathbf{u}_\varepsilon \lesssim \varepsilon^{-1/\sigma} |\mathbf{u}|_{\ell_w^p(\Lambda)}^{1/\sigma},$$

$$(II) \quad \text{the number of arithmetic operations to compute } \mathbf{u}_\varepsilon \text{ is at most a multiple of } \varepsilon^{-1/\sigma} |\mathbf{u}|_{\ell_w^p(\Lambda)}^{1/\sigma}.$$

Furthermore, $\|\mathbf{P}\mathbf{u} - \mathbf{u}_\varepsilon\|_{\ell^2(\Lambda)} \leq \varepsilon$ and so $\|u - G_\Phi^* D_B^{-1} \mathbf{u}_\varepsilon\|_{\mathcal{H}} \lesssim \varepsilon$.

An analogous result holds also for the algorithm **SOLVE**, provided that the orthogonal projector \mathbf{P} onto the range of \mathbf{F} is σ^* -compressible as above. However, except for trivial cases it is not possible to verify this assumption with current mathematical technology [Ste04, DFR07].

The line of attack to solve the operator equation (2.1) is now clear: We have to construct a Gelfand frame Φ for the Gelfand triple $(H^{\vec{s}}, L^2, (H^{\vec{s}})')$ and show that the resulting matrices \mathbf{F} and \mathbf{P} are compressible. This is done in the following sections.

4 Ridgelet Frames

4.1 Ridgelet Gelfand Frames for $H^{\vec{s}}$

In order to make use of the general results of the previous subsection for our problem (2.6), leading to a stable discretization, the task is to construct a Gelfand frame for the Gelfand triple induced by $\mathcal{H} = H^{\vec{s}}(\mathbb{R}^d)$.

To this end, in [Gro11], a Parseval frame $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda}$ of ridgelets was constructed – we need to reproduce it in some detail, in order to be able to derive a number of properties which will be indispensable to prove sparsity of the operator in this discretisation. The key to the construction is a certain set of functions $\psi_{j,\ell} \in L^2(\mathbb{R}^d)$, which form a partition of unity in the frequency domain, i.e.

$$(\psi_{j,\ell})_{j \in \mathbb{N}_0, \ell \in \{0, \dots, L_j\}} \quad \text{such that} \quad \sum_{j=0}^{\infty} \sum_{\ell=0}^{L_j} \hat{\psi}_{j,\ell}^2 = 1. \quad (4.1)$$

Definition 4.1. To partition the angular component, we need a covering (approximately uniform) of the sphere \mathbb{S}^{d-1} , which we choose according to the following construction for $\alpha = 2^{-j}$:

$$\text{Choose } \{\vec{s}_\ell \in \mathbb{S}^{d-1}\}_{\ell \in \{0, \dots, L\}} \quad \text{such that} \quad \begin{cases} \bigcup_{\ell=0}^L B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \alpha) = \mathbb{S}^{d-1}, \\ B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \frac{\alpha}{3}) \text{ pairwise disjoint.} \end{cases}$$

Here $B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha)$ is the open ball on the sphere of radius α in the geodesic metric (see Subsection A.1). This can be shown to imply $L \sim (\frac{1}{\alpha})^{d-1}$ (for details see [BN07] or Subsection A.2), and thus $L_j \sim 2^{j(d-1)}$.

Furthermore, set $\alpha_j := 2^{-j+1}$ and choose (smooth and bounded) window functions W , $W^{(0)}$, $V^{(j,\ell)}$, such that

1. $\text{supp } W \subseteq (\frac{1}{2}, 2)$,
2. $\text{supp } W^{(0)} \subseteq [0, 2)$,
3. $\text{supp } V^{(j,\ell)} \subseteq B_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \alpha_j)$,
4. Lower bounds for all functions in a suitable subset, see [Gro11].

From these properties, it can be shown that

$$\Phi(\vec{\xi}) := W^{(0)}(|\vec{\xi}|)^2 + \sum_{j \in \mathbb{N}_0} \sum_{\ell=0}^{L_j} W(2^{-j}|\vec{\xi}|)^2 V^{(j,\ell)}\left(\frac{\vec{\xi}}{|\vec{\xi}|}\right)^2$$

is bounded from above and below. Now, define

$$\hat{\psi}_{0,0}(\vec{\xi}) := \frac{W^{(0)}(|\vec{\xi}|)}{\sqrt{\Phi(\vec{\xi})}} \quad \text{and} \quad \hat{\psi}_{j,\ell}(\vec{\xi}) := \frac{W(2^{-j}|\vec{\xi}|) V^{(j,\ell)}\left(\frac{\vec{\xi}}{|\vec{\xi}|}\right)}{\sqrt{\Phi(\vec{\xi})}}, \quad j \geq 1, \ell = 0, \dots, L_j. \quad (4.2)$$

Note that all $\psi_{j,\ell}$ are defined via their Fourier transforms, and that for notational convenience, we have set $L_0 := 0$ to extend the indexing consistently to the function for $j = 0$ as well. From these definitions, it is easy to check that (4.1) holds.

Definition 4.2. Using Definition 4.1, a Parseval frame for $L^2(\mathbb{R}^d)$ is defined by

$$\varphi_{j,\ell,\vec{k}} = 2^{-\frac{j}{2}} T_{U_{j,\ell}\vec{k}} \psi_{j,\ell}, \quad j \in \mathbb{N}_0, \ell \in \{0, \dots, L_j\}, \vec{k} \in \mathbb{Z}^d,$$

with T the translation operator, $T_{\vec{y}}f(\cdot) := f(\cdot - \vec{y})$, and $U_{j,\ell} := R_{j,\ell}^{-1} D_{2^{-j}}$, where $R_{j,\ell}$ is the transformation introduced in Section 2, and D_a dilates the first component, $D_a \vec{k} := (a k_1, k_2, \dots, k_d)^\top$. The rotation $R_{j,\ell}$ is arbitrary (to the extent that it is ambiguous, see Remark 4.6) but fixed. Whenever possible, we will subsume the indices of φ by $\lambda = (j, \ell, \vec{k})$.

We note that for a Parseval frame, the frame operator $S_\Phi = \mathbb{I}$, since

$$\langle S_\Phi f, f \rangle = \langle G^* G f, f \rangle = \langle G f, G f \rangle = \|\langle \Phi, f \rangle\|_{\ell^2}^2 = \|f\|^2 = \langle f, f \rangle,$$

which implies $\tilde{\Phi} = \Phi$.

With the ridgelet frame Φ in hand we go on to show that Φ is indeed a Gelfand frame for the Gelfand triple $(H^{\vec{s}}, L^2(\mathbb{R}^d), (H^{\vec{s}})')$. First, we need to find suitable sequence spaces \mathcal{H}_d . To this end we introduce the diagonal preconditioning matrix

$$\mathbf{W}_{\lambda,\lambda'} = \begin{cases} 0, & \lambda \neq \lambda', \\ w(\lambda) := 1 + 2^j |\vec{s} \cdot \vec{s}_{j,\ell}|, & \lambda = \lambda', \end{cases}$$

and define the weighted ℓ^2 -spaces

$$\mathcal{H}_d := \ell_{\mathbf{W}}^2(\Lambda) := \{\mathbf{c} \in \ell^2(\Lambda) : \|\mathbf{W}\mathbf{c}\|_{\ell^2(\Lambda)} < \infty\}$$

and the corresponding isomorphisms

$$D_{\ell^2, \mathbf{W}} : \begin{cases} \mathcal{H}_d & \rightarrow \ell^2(\Lambda), \\ \mathbf{c} & \mapsto \mathbf{W}\mathbf{c}, \end{cases} \quad \text{and} \quad D_{\ell^2, \mathbf{W}}^* : \begin{cases} \ell^2(\Lambda) & \rightarrow \mathcal{H}_d' = \ell_{\mathbf{W}^{-1}}^2(\Lambda), \\ \mathbf{c} & \mapsto \mathbf{W}\mathbf{c}. \end{cases}$$

Theorem 4.3. *The ridgelet frame Φ as constructed above constitutes a Gelfand frame for the Gelfand triple $(H^{\vec{s}}, L^2(\mathbb{R}^d), (H^{\vec{s}})')$.*

Proof. Essentially, this has been shown in [Gro11], where it is observed that

$$\|f\|_{H^{\vec{s}}} \sim \|\langle \Phi, f \rangle_{\mathcal{H}}\|_{\ell_{\mathbf{W}}^2}. \quad (4.3)$$

Using the fact that $\tilde{\Phi} = \Phi$, we immediately infer boundedness of the operator $G_{\tilde{\Phi}} : H^{\vec{s}} \rightarrow \ell_{\mathbf{W}}^2(\Lambda)$. To show the boundedness of the operator $G_{\Phi}^* : \ell_{\mathbf{W}}^2(\Lambda) \rightarrow H^{\vec{s}}$ we need to estimate the $H^{\vec{s}}$ -norm of $\Phi \mathbf{c}$ in terms of the $\ell_{\mathbf{W}}^2$ -norm of \mathbf{c} . To see this, we first observe that, due to (4.3), we have

$$\|\Phi \mathbf{c}\|_{H^{\vec{s}}} \lesssim \|\langle \Phi, \Phi \rangle_{L^2} \mathbf{c}\|_{\ell_{\mathbf{W}}^2}.$$

In order to arrive at the desired bound it remains to note that the matrix operator $\langle \Phi, \Phi \rangle_{L^2} : \ell_{\mathbf{W}}^2(\Lambda) \rightarrow \ell_{\mathbf{W}}^2(\Lambda)$ is bounded, which is a simple consequence of the frequency support properties of the frame elements. \square

Theorem 4.4. *With Φ the ridgelet system and A the differential operator defined above, consider the (infinite) matrix*

$$\mathbf{F} := \mathbf{W}^{-1} \langle A \Phi, A \Phi \rangle_{L^2} \mathbf{W}^{-1}. \quad (4.4)$$

Then the operator $\mathbf{F} : \ell^2(\Lambda) \rightarrow \ell^2(\Lambda)$ is bounded as well as boundedly invertible on its range $\text{ran}(\mathbf{F}) = \text{ran}((D_{\ell_{\mathbf{W}}^2})^{-1} G_{\Phi})$.

Proof. This is a direct consequence of Lemma 3.4, Corollary 2.5 and Theorem 4.3. \square

In summary, we have achieved (i) above, namely a stable discretization of the operator equation (2.6). In order to make use of the convergence results presented in Section 3 we also need to derive a matrix representation of the projector \mathbf{P} defined in subsubsection 3.2.1. Using the fact that for our ridgelet frame construction Φ , the dual frame coincides with the primal frame, e.g., $\tilde{\Phi} = \Phi$, it is easy to see that

$$\mathbf{P} = \mathbf{W} \langle \Phi, \Phi \rangle_{L^2} \mathbf{W}^{-1}. \quad (4.5)$$

4.2 Remarks on the construction

Remark 4.5. By the support properties of $V^{(j,\ell)}$ and W , we see that

$$\begin{aligned} \text{supp } \hat{\psi}_{j,\ell} &\subseteq P_{j,\ell} := \left\{ \vec{\xi} \in \mathbb{R}^d : 2^{j-1} < |\vec{\xi}| < 2^{j+1}, \frac{\vec{\xi}}{|\vec{\xi}|} \in B_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \alpha_j) \right\}, \quad j \geq 1, \\ \text{supp } \hat{\psi}_{0,0} &\subseteq P_{0,0} := \left\{ \vec{\xi} \in \mathbb{R}^d : |\vec{\xi}| < 2 \right\}. \end{aligned}$$

For several reasons, we will need to know for which j and ℓ the intersections $P_{j,\ell} \cap P_{j',\ell'}$ are non-empty if j', ℓ' is fixed. For example, to prove the sparsity of the ridgelet discretisation of the transport operator A introduced in Section 2, we will have to consider a sum of terms involving the $\hat{\psi}_{j,\ell}$ over all parameters as in (5.1) – only through a criterion of the above-mentioned form will we be able to bound the sum. Luckily, it is straightforward to check that a non-empty intersection necessarily implies

$$|j - j'| \leq 1 \quad \text{and} \quad \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}) \leq \alpha_j + \alpha_{j'} \stackrel{(*)}{\leq} 3\alpha_{j'},$$

where $(*)$ makes use of the first condition. Often, it will turn out to be convenient to cast these conditions into an inclusion, in other words,

$$\{(j, \ell) : P_{j,\ell} \cap P_{j',\ell'} \neq \emptyset\} \subseteq \{(j, \ell) : |j - j'| \leq 1, \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}) \leq 3\alpha_{j'}\}.$$

This knowledge lets us revisit the function Φ in Definition 4.1 – in particular, for $\vec{\xi} \in P_{j,\ell}$, the sum consists of only the terms “neighbouring” j and ℓ ,

$$\Phi(\vec{\xi}) = \sum_{\substack{j' \in \mathbb{N}_0 : \\ |j - j'| \leq 1}} \sum_{\substack{\ell' \in \{0, \dots, L_{j'}\} : \\ \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}) \leq 3\alpha_j}} W(2^{-j'} |\vec{\xi}|)^2 V^{(j',\ell')} \left(\frac{\vec{\xi}}{|\vec{\xi}|} \right)^2. \quad (4.6)$$

Of course, the above describes the case $j > 2$ – otherwise, the term $W^{(0)}(|\vec{\xi}|)$ would also appear in the sum.

Remark 4.6. In dimensions $d > 3$, the rotation $R_{\vec{s}}$ turning \vec{s} into \vec{e}_1 is no longer unique, although all other possible choices must satisfy

$$\widetilde{R_{\vec{s}}} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} R_{\vec{s}},$$

where $R \in \text{SO}(d-1)$. Due to this ambiguity, the Lipschitz condition

$$\|R_{\vec{s}} - R_{\vec{s}'}\| \lesssim \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}, \vec{s}') \quad (4.7)$$

will not hold in general (if, for example, $\vec{s} = \vec{s}'$ and the matrix R above contains two reflections). However, it is possible to choose such an $R_{\vec{s}}$ for fixed $R_{\vec{s}'}$ (as proved in Lemma A.4) – this suffices for our purposes, since in essence, we do not need this Lipschitz condition globally, but only in a neighbourhood of \vec{s} , where the ambiguity is irrelevant.

In the course of the proof (of sparsity of the Ridgelet discretisation), we need to control the derivatives of $\hat{\psi}_{j,\ell}$ under a pullback related to the above-mentioned $U_{j,\ell}$. We formulate this as an assumption that has to be satisfied when choosing the window functions.

Assumption 4.7. *The window functions in Definition 4.1 are chosen in such a way, that for any rotation $R_{j,\ell}$ (taking $\vec{s}_{j,\ell}$ to \vec{e}_1), the pullbacks under the transformation $U_{j,\ell}^{-\top} = R_{j,\ell}^{-1} D_{2^j}$,*

$$\hat{\psi}_{(j,\ell)}(\vec{\eta}) := \hat{\psi}_{j,\ell}(U_{j,\ell}^{-\top} \vec{\eta}) = \frac{W(2^{-j}|D_{2^j} \vec{\eta}|) V^{(j,\ell)}\left(\frac{D_{2^j} \vec{\eta}}{|D_{2^j} \vec{\eta}|}\right)}{\sqrt{\Phi(U_{j,\ell}^{-\top} \vec{\eta})}},$$

have bounded derivatives independently of j and ℓ . Thus, for all n up to an upper bound N dependent on the differentiability of the window functions (or possibly for all $n \in \mathbb{N}$ if the window functions are C^∞), we have the estimate

$$\|\hat{\psi}_{(j,\ell)}\|_{C^n} \leq \beta_n.$$

In Lemma B.1, we show that this assumption can be satisfied with a reasonable (and still quite flexible) choice of window functions.

5 Compressibility

In this section, we show the main result, that the relevant bi-infinite matrices (\mathbf{F} and \mathbf{P}) appearing in Algorithm 3.6 are in fact compressible.

5.1 Preliminary Considerations

In general, compressibility is difficult to verify directly. Instead we use the following notion of sparsity for a (possible bi-infinite) matrix \mathbf{A} :

Definition 5.1. Let $p > 0$. A matrix $\mathbf{A} = (a_{\lambda,\lambda'})_{\lambda \in \Lambda, \lambda' \in \Lambda'}$ is called p -sparse if

$$\|\mathbf{A}\|_{\ell^p(\Lambda) \rightarrow \ell^p(\Lambda')} := \max \left(\sup_{\lambda' \in \Lambda'} \sum_{\lambda \in \Lambda} |a_{\lambda,\lambda'}|^p, \sup_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda'} |a_{\lambda,\lambda'}|^p \right)^{\frac{1}{p}} < \infty. \quad (5.1)$$

Proposition 5.2. *Assume that \mathbf{A} is p -sparse for $0 < p < 1$. Then \mathbf{A} is $\frac{1}{2}(\frac{1}{p} - 1)$ -compressible.*

To prove this result, we require the following version of Schur's test (which is [HS78, Thm. 5.2] for a discrete measure):

Theorem 5.3. Let $\mathbf{A} := (a_{\lambda, \lambda'})_{\lambda, \lambda' \in \Lambda}$ be an operator. Then the following holds:

$$\|\mathbf{A}\|_{\ell^2(\Lambda) \rightarrow \ell^2(\Lambda)} \leq \left(\sup_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda'} |a_{\lambda, \lambda'}| \right)^{\frac{1}{2}} \left(\sup_{\lambda' \in \Lambda'} \sum_{\lambda \in \Lambda} |a_{\lambda, \lambda'}| \right)^{\frac{1}{2}}$$

Proof of Proposition 5.2. Note that by assumption each column \mathbf{a}_λ of \mathbf{A} has ℓ^p norm bounded by

$$\|\mathbf{a}_\lambda\|_p \leq \|\mathbf{A}\|_{\ell^p(\Lambda) \rightarrow \ell^p(\Lambda)}.$$

This means that for each $k \in \mathbb{N}$ and for some summable sequence α_k , we may keep only the $\alpha_k 2^k$ largest coefficients of the column vector \mathbf{a}_λ , which gives the approximation $\mathbf{A}^{[k]}$ consisting of columns $\mathbf{a}_\lambda^{[k]}$. An immediate observation is, that it would be non-sensical to let α_k decay quicker than 2^{-k} , since then the number of approximating coefficients would decrease in each step. While it is possible to let α_k decay like $2^{-k(1-\epsilon)}$, this would impact the achieved compressibility (see below), and so we choose a sequence whose inverses grow at most polynomially (say, $\alpha_k = k^{-2}$).

To compute the error of this approximation,

$$\|\mathbf{A} - \mathbf{A}^{[k]}\|_{\ell^2(\Lambda) \rightarrow \ell^2(\Lambda)},$$

denote by $\mathbf{a}_\lambda^* := ((a_\lambda^*)_i)_{i \in \mathbb{N}}$ the non-increasing rearrangement of \mathbf{a}_λ . The defining condition for weak- ℓ^p spaces is, that $\mathbf{a}_\lambda \in \ell^p$ implies $(a_\lambda^*)_i \lesssim i^{-\frac{1}{p}}$. In order to be able to apply Schur's Lemma, we use this fact to estimate the (square of the) first factor,

$$\sup_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} |a_{\lambda, \lambda'} - a_{\lambda, \lambda'}^{[k]}| = \sup_{\lambda \in \Lambda} \sum_{i \geq \alpha_k 2^k} |(a_\lambda^*)_i| \lesssim \sum_{i \geq \alpha_k 2^k} i^{-\frac{1}{p}}.$$

To continue, choose $\ell \in \mathbb{N}$ such that $2^{\ell-1} \leq \alpha_k 2^k \leq 2^\ell$ and let

$$F_m := \sum_{i=2^{m-1}}^{2^m} i^{-\frac{1}{p}} \leq (2^m - 2^{m-1})(2^{m-1})^{-\frac{1}{p}}.$$

Consequently, using $p < 1$, we see that

$$\begin{aligned} \sum_{i \geq \alpha_k 2^k} i^{-\frac{1}{p}} &\leq \sum_{i \geq 2^{\ell-1}} i^{-\frac{1}{p}} = \sum_{m=\ell}^{\infty} F_m \leq \sum_{m=\ell}^{\infty} 2^{-(m-1)(\frac{1}{p}-1)} = 2^{-(\ell-1)(\frac{1}{p}-1)} \sum_{m=0}^{\infty} 2^{-(m-1)(\frac{1}{p}-1)} \\ &= \left(\frac{1}{2^\ell}\right)^{\frac{1}{p}-1} \frac{2^{\frac{1}{p}-1}}{1 - 2^{-(\frac{1}{p}-1)}} \lesssim (\alpha_k 2^k)^{-(\frac{1}{p}-1)}. \end{aligned}$$

On the other hand, since all sequences \mathbf{x} satisfy $\|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p$ for all $0 < p \leq q \leq \infty$, we have (with $q = 1$)

$$\sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} |a_{\lambda, \lambda'} - a_{\lambda, \lambda'}^{[k]}| \leq 2 \sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} |a_{\lambda, \lambda'}| = 2 \sup_{\lambda' \in \Lambda} \|\mathbf{a}_{\lambda'}\|_1 \leq 2 \sup_{\lambda' \in \Lambda} \|\mathbf{a}_{\lambda'}\|_p \leq 2 \|\mathbf{A}\|_{\ell^p(\Lambda) \rightarrow \ell^p(\Lambda)} < \infty.$$

Applying Schur's Lemma we get that

$$\|\mathbf{A} - \mathbf{A}^{[k]}\|_{\ell^2(\Lambda) \rightarrow \ell^2(\Lambda)} \lesssim (\alpha_k^{-1} 2^{-k})^{\frac{1}{2}(\frac{1}{p}-1)} =: C_k.$$

We see that for any $\sigma < \frac{1}{2}(\frac{1}{p}-1)$, the sequence $(2^{\sigma k} C_k)_{k \in \mathbb{N}}$ is summable, since the polynomial growth in α_k^{-1} does not affect the exponential decay of $2^{-(\frac{1}{2}(\frac{1}{p}-1)-\sigma)k}$ (up to a constant). This proves the compressibility. \square

5.2 Sparsity of \mathbf{F}

Having introduced the concepts of compressibility and sparsity in the last section, we now want to show sparsity of the ridgelet discretisation of the transport operator A introduced in Section 2. By Proposition 5.2, this will prove compressibility of the ridgelet discretisation of this operator.

Theorem 5.4. *We consider the frame $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda}$ for $L^2(\mathbb{R}^d)$ (see Definition 4.2), satisfying Assumption 4.7 for $2n$ with $\frac{d}{2} < n \in \mathbb{N}$, and choose $p \in \mathbb{R}$ such that $1 > p > \frac{d}{2n}$. Furthermore, we introduce the differential operator $A : u \mapsto \vec{s} \cdot \nabla u + \kappa u$ with fixed $\vec{s} \in \mathbb{S}^{d-1}$, where the absorption coefficient κ has a decomposition $\kappa = \gamma + \kappa_0$ with constant $\gamma > 0$, and $\kappa_0 \geq 0$ satisfying $\kappa_0, \hat{\kappa}_0 \in L_\infty(\mathbb{R}^d)$. Finally, we demand the existence of $r_0, c_0 > 0$, such that the decay condition*

$$|\hat{\kappa}_0(\vec{\xi})| \leq \frac{c_0}{|\vec{\xi}|^q} \quad \forall \vec{\xi} \in \mathbb{R}^d : |\vec{\xi}| \geq r_0, \quad (5.2)$$

is fulfilled for a fixed $q > 2d + 2n + \frac{3}{2} + \frac{d-1}{p}$. Then the preconditioned stiffness matrix \mathbf{F} , see (4.4), is p -sparse in this frame – in other words,

$$\|\mathbf{F}\|_{\ell^p(\Lambda) \rightarrow \ell^p(\Lambda)} = \left\| \mathbf{W}^{-1} \langle A \Phi, A \Phi \rangle_{L^2} \mathbf{W}^{-1} \right\|_{\ell^p(\Lambda) \rightarrow \ell^p(\Lambda)} < \infty. \quad (5.3)$$

Remark 5.5. As we have seen in Proposition 5.2, the smaller p , the better the compressibility. The theorem is formulated in a way that p is chosen according to the restrictions imposed by d and n – however, since it is possible to construct window functions of arbitrary smoothness (and thus arbitrarily smooth $\psi_{j,\ell}$), the limiting factor for p then becomes the decay rate of $\hat{\kappa}_0$. In the case that $\hat{\kappa}_0$ decays faster than any polynomial (say, exponentially), arbitrarily small p can be achieved (for infinitely smooth $\psi_{j,\ell}$) – of course at the cost of exploding constants.

Before we begin with the proof, as a service to the reader, we collect a few results on technical details, which we have moved to Appendix A.

Proposition 5.6 (Lemma A.5). *Let $w(\lambda) = 1 + 2^j |\vec{s} \cdot \vec{s}_{j,\ell}|$, $U_{j,\ell} = R_{j,\ell}^{-1} D_{2^{-j}}$ and $U_{j',\ell'} = R_{j',\ell'}^{-1} D_{2^{-j'}}$ with arbitrary $R_{j',\ell'}$ such that (4.7) holds for $\vec{s} = \vec{s}_{j,\ell}$ and $\vec{s}' = \vec{s}_{j',\ell'}$. Then we have the estimates*

$$|U_{j,\ell}^{-1} \vec{s}| \leq w(\lambda), \quad \text{and} \quad |U_{j,\ell}^{-1} \vec{s}| \lesssim \max(2^{j-j'}, 1) (w(\lambda') + 2^{j'} \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'})), \quad (5.4)$$

as well as

$$\|U_{j,\ell}^{-1} U_{j',\ell'}\| \lesssim \max(2^{j-j'}, 1) + 2^j \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}). \quad (5.5)$$

Proposition 5.7 (Proposition A.6). *For $j \geq 1$, the transformation $U_{j,\ell}^\top$ takes the “frequency tiles” $P_{j,\ell}$ back into a bounded set around the origin (illustrated in Figure A.1),*

$$U_{j,\ell}^\top P_{j,\ell} \subseteq B_{\mathbb{R}^d}(0, 5) \quad \text{and} \quad U_{j,\ell}^\top (P_{j,\ell}^m) \subseteq B_{\mathbb{R}^d}(0, 5 + 2^m), \quad (5.6)$$

where $P_{j,\ell}^m$ is again the Minkowski sum $P_{j,\ell} + B_{\mathbb{R}^d}(0, 2^m)$.

Additionally, we can calculate the opening angle of the cone containing $P_{j,\ell}^m$ as follows,

$$\alpha_j^m = \alpha_j + \arcsin\left(\frac{2^m}{2^{j-1}}\right) \leq c_\omega 2^{m-j}, \quad (5.7)$$

as long as $j \geq m + 1$, where $c_\omega \leq \pi + 2$ (illustrated in Figure A.2).

Proposition 5.8 (Lemma A.7). *Let j', ℓ' as well as m, m' be fixed and denote $m_{>} := \max(m, m')$, then we have the following inclusion for the set of parameters that can yield a non-empty intersection with $P_{j',\ell'}^{m'}$,*

$$\{(j, \ell) : P_{j,\ell}^m \cap P_{j',\ell'}^{m'} \neq \emptyset\} \subseteq \bigcup_{j=0}^{m_{>}+2} \{j\} \times \{0 \leq \ell \leq L_j\} \cup \bigcup_{j \geq m_{>}+3} \{j\} \times \{\ell : \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}) \leq 5c_\omega 2^{m_{>}-j'}\}, \quad (5.8)$$

Proof of Theorem 5.4. Due to symmetry, we are able to express (5.3) without taking the maximum (cf. (5.1)),

$$\begin{aligned} \|\mathbf{F}\|_{\ell^p(\Lambda) \rightarrow \ell^p(\Lambda)}^p &= \sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} \left| w(\lambda)^{-1} w(\lambda')^{-1} \langle A \varphi_\lambda, A \varphi_{\lambda'} \rangle_{L^2} \right|^p \\ &= \sup_{\lambda' \in \Lambda} \sum_{j \in \mathbb{N}_0} \sum_{\ell=0}^{L_j} \sum_{\vec{k} \in \mathbb{Z}^d} \left| w(\lambda)^{-1} w(\lambda')^{-1} \langle A \varphi_{j,\ell,\vec{k}}, A \varphi_{j',\ell',\vec{k}'} \rangle_{L^2} \right|^p < \infty. \end{aligned} \quad (5.9)$$

Step 1 – Transforming the integral: Recalling the definition of the φ_λ , we compute

$$\begin{aligned} \mathbf{F}_{\lambda,\lambda'} &= w(\lambda)^{-1} w(\lambda')^{-1} \langle A \varphi_\lambda, A \varphi_{\lambda'} \rangle_{L^2} = w(\lambda)^{-1} w(\lambda')^{-1} \left\langle \widehat{A \varphi_\lambda}, \widehat{A \varphi_{\lambda'}} \right\rangle_{L^2} \\ &= \frac{2^{-\frac{j+j'}{2}}}{w(\lambda)w(\lambda')} \int \overline{\mathcal{F}(\vec{s} \cdot \nabla \psi_{j,\ell}(\vec{x} - U_{j,\ell} \vec{k}) + \kappa(\vec{x}) \psi_{j,\ell}(\vec{x} - U_{j,\ell} \vec{k}))(\vec{\xi})} \dots \\ &\quad \dots \mathcal{F}(\vec{s} \cdot \nabla \psi_{j',\ell'}(\vec{x} - U_{j',\ell'} \vec{k}') + \kappa(\vec{x}) \psi_{j',\ell'}(\vec{x} - U_{j',\ell'} \vec{k}'))(\vec{\xi}) \, d\vec{\xi} \\ &= \frac{c_{\mathcal{F}} 2^{-\frac{j+j'}{2}}}{w(\lambda)w(\lambda')} \int \overline{((\vec{s} \cdot \vec{\xi}) \hat{\psi}_{j,\ell}(\vec{\xi}) + [\hat{\psi}_{j,\ell} * M_\lambda \hat{\kappa}](\vec{\xi}))} \dots \\ &\quad \dots ((\vec{s} \cdot \vec{\xi}) \hat{\psi}_{j',\ell'}(\vec{\xi}) + [\hat{\psi}_{j',\ell'} * M_{\lambda'} \hat{\kappa}](\vec{\xi})) \exp(2\pi i \vec{\xi} \cdot (U_{j,\ell} \vec{k} - U_{j',\ell'} \vec{k}')) \, d\vec{\xi}, \end{aligned}$$

where $c_{\mathcal{F}} = (2\pi i)^2$ and $[M_\lambda \hat{\kappa}](\vec{\xi}) = \hat{\kappa}(\vec{\xi}) \exp(2\pi i \vec{\xi} \cdot U_{j,\ell} \vec{k}) = \mathcal{F}(\kappa(\vec{x} + U_{j,\ell} \vec{k}))(\vec{\xi})$ is the modulation operator corresponding to a shift of $U_{j,\ell} \vec{k}$ according to our definition of the Fourier transform.

The transformation $U_{j,\ell}$ modifying \vec{k} in the exponential function makes summing \vec{k} difficult, and therefore, we will transform all the integral by $\vec{\xi} = U_{j,\ell}^{-\top} \vec{\eta}$ – introducing a factor 2^j from the determinant of the Jacobian and yielding the exponent

$$2\pi i \vec{\eta} \cdot (\vec{k} - U_{j,\ell}^{-1} U_{j',\ell'} \vec{k}') = 2\pi i \vec{\eta} \cdot (\vec{k} - U_{j',\ell'}^{j,\ell} \vec{k}'), \quad \text{where} \quad U_{j',\ell'}^{j,\ell} := U_{j,\ell}^{-1} U_{j',\ell'}.$$

Additionally, we observe that the Fourier transform of κ splits as follows, $\hat{\kappa}(\vec{\xi}) = (\widehat{\gamma + \kappa_0})(\vec{\xi}) = \gamma \delta(\vec{\xi}) + \hat{\kappa}_0(\vec{\xi})$. We expand all products and rearrange, observing also that $\vec{s} \cdot \vec{\xi} = \vec{s} \cdot U_{j,\ell}^{-\top} \vec{\eta} = U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta}$ is a real number. Thus,

$$\begin{aligned} \mathbf{F}_{\lambda,\lambda'} &= \frac{c_{\mathcal{F}} 2^{\frac{j-j'}{2}}}{w(\lambda)w(\lambda')} \int \left((U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma)^2 \underbrace{\hat{\psi}_{(j,\ell)}(\vec{\eta}) \hat{\psi}_{(j',\ell')}(\vec{\eta}) (\tilde{U}_{j',\ell'}^\top U_{j,\ell}^{-\top} \vec{\eta})}_{=: h_{\lambda,\lambda'}^{00}(\vec{\eta})} + \dots \right. \\ &\quad \dots (U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma) \underbrace{\psi_{(j,\ell)}(\vec{\eta}) [\hat{\psi}_{j',\ell'} * M_{\lambda'} \hat{\kappa}_0](U_{j,\ell}^{-\top} \vec{\eta})}_{=: h_{\lambda,\lambda'}^{0*}(\vec{\eta})} + \dots \\ &\quad \dots (U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma) \underbrace{[\hat{\psi}_{j,\ell} * M_\lambda \hat{\kappa}_0](U_{j,\ell}^{-\top} \vec{\eta}) \psi_{(j',\ell')}(\vec{\eta}) (\tilde{U}_{j',\ell'}^\top U_{j,\ell}^{-\top} \vec{\eta})}_{=: h_{\lambda,\lambda'}^{*0}(\vec{\eta})} + \dots \\ &\quad \left. \dots \underbrace{[\hat{\psi}_{j,\ell} * M_\lambda \hat{\kappa}_0](U_{j,\ell}^{-\top} \vec{\eta}) [\hat{\psi}_{j',\ell'} * M_{\lambda'} \hat{\kappa}_0](U_{j,\ell}^{-\top} \vec{\eta})}_{=: h_{\lambda,\lambda'}^{**}(\vec{\eta})} \right) \exp(2\pi i \vec{\eta} \cdot (\vec{k} - U_{j',\ell'}^{j,\ell} \vec{k}')) \, d\vec{\eta}, \quad (5.10) \end{aligned}$$

where we used the representation of $\hat{\psi}_{j,\ell}$ from Assumption 4.7 – which holds for arbitrary rotations $\tilde{R}_{j',\ell'}$ taking $\vec{s}_{j',\ell'}$ to \vec{e}_1 . We choose $\tilde{R}_{j',\ell'}$ in $\tilde{U}_{j',\ell'} := \tilde{R}_{j',\ell'}^{-1} D_{2^{-j}}$ in such a way that (4.7) holds for $\vec{s} = \vec{s}_{j,\ell}$ and $\vec{s}' = \vec{s}_{j',\ell'}$, which is possible due to Lemma A.4. Unsurprisingly, we set $\tilde{U}_{j',\ell'}^{j,\ell} := U_{j,\ell}^{-1} \tilde{U}_{j',\ell'}$.

It should be noted that h -terms does not depend on \vec{k}, \vec{k}' – however, we have chosen this notation for reasons of readability (as well as uniformity with the Y - and Z -terms, which appear in Step 2 and 7, respectively).

For each of the terms in (5.10), we have to show that the sum over all parameters in (5.9) is finite – which we will do for \vec{k} first, then for ℓ and finally for j .

Step 2 – Integration by parts: Even though the exponent is purely imaginary, we cannot estimate the exponential function by one, as we would then sum constants in \vec{k} . However, a simple calculation shows $\Delta_{\vec{\eta}} \exp(2\pi i \vec{\eta} \cdot \vec{y}) = -(2\pi)^2 |\vec{y}|^2 \exp(2\pi i \vec{\eta} \cdot \vec{y})$, which entails

$$\Delta_{\vec{\eta}} \exp(2\pi i \vec{\eta} \cdot (\vec{k} - U_{j,\ell}^{j,\ell} \vec{k}')) = -(2\pi)^2 |\vec{k} - U_{j,\ell}^{j,\ell} \vec{k}'|^2 \exp(2\pi i \vec{\eta} \cdot (\vec{k} - U_{j,\ell}^{j,\ell} \vec{k}')).$$

Applying Green's second identity iteratively, we will use this to generate a denominator of sufficient power to be summed over all $\vec{k} \in \mathbb{Z}^d$ – on the other hand, this forces us to estimate the derivatives of the remaining factors of the integrands. All differential operators will be with respect to $\vec{\eta}$, which we will not indicate anymore in the following.

The $h_{\lambda,\lambda'}^{00}$ is unproblematic because of its compact support, however, for the other functions, their unbounded support (as a superset of $\text{supp } \hat{\kappa}_0$) means that the boundary integral does not vanish trivially. Nevertheless, we can always shift the derivatives of the convolution away from $\hat{\kappa}$ and so the vanishing of the boundary term can be seen by applying Green's second identity to the domain $B_{\mathbb{R}^d}(0, R)$ and exploiting the decay of $\hat{\kappa}$ as $R \rightarrow \infty$ (bearing in mind that λ, λ' are fixed for this consideration). Due to the growing surface of $B_{\mathbb{R}^d}(0, R)$, this requires a decay $q > d - 1$ of $\hat{\kappa}_0$, which is satisfied by our assumption.

Thus, for $\vec{k} \neq U_{j,\ell}^{j,\ell} \vec{k}'$,

$$\begin{aligned} \mathbf{F}_{\lambda,\lambda'} &= \frac{c_{\mathcal{F}} 2^{\frac{j-j'}{2}}}{w(\lambda)w(\lambda')} \frac{(-1)^n (2\pi)^{-2n}}{|\vec{k} - U_{j,\ell}^{j,\ell} \vec{k}'|^{2n}} \int \left[\Delta^n \left((U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma)^2 h_{\lambda,\lambda'}^{00}(\vec{\eta}) \right) + \Delta^n \left((U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma) h_{\lambda,\lambda'}^{0*}(\vec{\eta}) \right) + \dots \right. \\ &\quad \left. \dots + \Delta^n \left((U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma) h_{\lambda,\lambda'}^{*0}(\vec{\eta}) \right) + \Delta^n \left(h_{\lambda,\lambda'}^{**}(\vec{\eta}) \right) \right] \exp(2\pi i \vec{\eta} \cdot (\vec{k} - U_{j,\ell}^{j,\ell} \vec{k}')) d\vec{\eta} \\ &=: |\vec{k} - U_{j,\ell}^{j,\ell} \vec{k}'|^{-2n} \left(Y_{\lambda,\lambda'}^{00} + Y_{\lambda,\lambda'}^{0*} + Y_{\lambda,\lambda'}^{*0} + Y_{\lambda,\lambda'}^{**} \right). \end{aligned} \quad (5.11)$$

Step 3 – Estimating the Derivatives: Before we can deal with the derivatives of the h -terms, we have to disentangle the derivatives of $U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta}$ from them. Computing $\nabla(U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma) = U_{j,\ell}^{-1} \vec{s}$, a simple induction shows

$$\Delta^n((U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma) h(\vec{\eta})) = (U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma) \Delta^n h(\vec{\eta}) + 2n U_{j,\ell}^{-1} \vec{s} \cdot \nabla(\Delta^{n-1} h(\vec{\eta})), \quad (5.12)$$

and inserting $\tilde{h}(\vec{\eta}) := (U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma) h(\vec{\eta})$ into this formula also yields

$$\begin{aligned} \Delta^n((U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma)^2 h(\vec{\eta})) &= (U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma)^2 \Delta^n h(\vec{\eta}) + 4n (U_{j,\ell}^{-1} \vec{s} \cdot \vec{\eta} + \gamma) (U_{j,\ell}^{-1} \vec{s} \cdot \nabla(\Delta^{n-1} h(\vec{\eta}))) + \dots \\ &\quad \dots + 2n |U_{j,\ell}^{-1} \vec{s}|^2 \Delta^{n-1} h(\vec{\eta}) + 4n(n-1) U_{j,\ell}^{-1} \vec{s} \cdot \left(\frac{\partial^2}{\partial \eta_s \partial \eta_t} (\Delta^{n-2} h(\vec{\eta})) \right)_{s,t=1}^d U_{j,\ell}^{-1} \vec{s}. \end{aligned} \quad (5.13)$$

Alternatively, both of these formulas can be obtained by applying (C.1) – the product rule for the Laplacian. This product rule is also the tool to obtain the following estimate, see Corollary C.3,

$$|[\Delta^n(fg)](\vec{\eta})| \leq (4d)^n |f(\vec{\eta})|_{\mathcal{C}^{2n}} |g(\vec{\eta})|_{\mathcal{C}^{2n}} \leq (4d)^n \|f\|_{\mathcal{C}^{2n}} \|g\|_{\mathcal{C}^{2n}}, \quad (5.14)$$

where $|f(\vec{\eta})|_{\mathcal{C}^{2n}} = \max_{0 \leq r \leq 2n} |f^{(r)}(\vec{\eta})|$ is the maximum of all derivatives up to order $2n$ of f at $\vec{\eta}$. We will use this to estimate the derivatives of the h -terms.

The last ingredient of this step is an estimate of the derivatives of the convolution terms, proved in Lemma C.4,

$$\left| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} ((f * g)(U\vec{\eta})) \right| \leq \left\| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (f(U\cdot)) \right\|_\infty (\mathbb{1}_{\text{supp } f}(\cdot) * |g(\cdot)|)(U\vec{\eta}), \quad (5.15)$$

where U is any invertible linear transformation and $\frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha}$ is the standard differentiation in multi-index notation. The reason for the form of this estimate will become apparent in the next step.

Using Assumption 4.7 with an appropriate choice of $\tilde{R}_{j',\ell'}$ (see (4.7)), together with $\|\hat{\psi}_{(j,\ell)}\|_{\mathcal{C}^{2n}} \leq \beta_{2n} < \infty$, (5.15) yields

$$\begin{aligned} \left| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} ((\hat{\psi}_{j',\ell'} * M_{\lambda'} \hat{\kappa}_0)(U_{j,\ell}^{-\top} \vec{\eta})) \right| &\leq \left\| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (\hat{\psi}_{j',\ell'}(U_{j,\ell}^{-\top} \cdot)) \right\|_\infty (\mathbb{1}_{P_{j',\ell'}} * |M_{\lambda'} \hat{\kappa}_0|)(U_{j,\ell}^{-\top} \vec{\eta}) \\ &= \left\| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (\hat{\psi}_{(j',\ell')}((\tilde{U}_{j',\ell'}^{j,\ell})^\top \cdot)) \right\|_\infty (\mathbb{1}_{P_{j',\ell'}} * |\hat{\kappa}_0|)(U_{j,\ell}^{-\top} \vec{\eta}) \\ &\leq \beta_{|\alpha|} \|\tilde{U}_{j',\ell'}^{j,\ell}\|^{|\alpha|} (\mathbb{1}_{P_{j',\ell'}} * |\hat{\kappa}_0|)(U_{j,\ell}^{-\top} \vec{\eta}), \end{aligned} \quad (5.16)$$

as well as

$$\begin{aligned} \left| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} ((\hat{\psi}_{j,\ell} * M_\lambda \hat{\kappa}_0)(U_{j,\ell}^{-\top} \vec{\eta})) \right| &\leq \left\| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} \hat{\psi}_{(j,\ell)} \right\|_\infty (\mathbb{1}_{P_{j,\ell}} * |M_\lambda \hat{\kappa}_0|)(U_{j,\ell}^{-\top} \vec{\eta}) \\ &\leq \beta_{|\alpha|} (\mathbb{1}_{P_{j,\ell}} * |\hat{\kappa}_0|)(U_{j,\ell}^{-\top} \vec{\eta}). \end{aligned} \quad (5.17)$$

Equations (5.12)–(5.17) allow us to estimate the different terms in (5.11), remembering to keep the support information of terms we estimate away:

$$\begin{aligned} |Y_{\lambda,\lambda'}^{00}| &\lesssim 2^{\frac{j-j'}{2}} \int_{U_{j,\ell}^\top (P_{j,\ell} \cap P_{j',\ell'})} \|\tilde{U}_{j',\ell'}^{j,\ell}\|^{2n} \frac{|U_{j,\ell}^{-1} \vec{s}|^2}{w(\lambda)w(\lambda')} (|\vec{\eta}|^2 + (4n+2\gamma)|\vec{\eta}| + 4n(n+\gamma) + \gamma^2) d\vec{\eta} \\ |Y_{\lambda,\lambda'}^{0*}| &\lesssim 2^{\frac{j-j'}{2}} \int_{U_{j,\ell}^\top P_{j,\ell}} \|\tilde{U}_{j',\ell'}^{j,\ell}\|^{2n} \frac{|U_{j,\ell}^{-1} \vec{s}|}{w(\lambda)w(\lambda')} (|\vec{\eta}| + 2n + \gamma) (\mathbb{1}_{P_{j',\ell'}} * |\hat{\kappa}_0|)(U_{j,\ell}^{-\top} \vec{\eta}) d\vec{\eta} \\ |Y_{\lambda,\lambda'}^{*0}| &\lesssim 2^{\frac{j-j'}{2}} \int_{U_{j,\ell}^\top P_{j',\ell'}} \|\tilde{U}_{j',\ell'}^{j,\ell}\|^{2n} \frac{|U_{j,\ell}^{-1} \vec{s}|}{w(\lambda)w(\lambda')} (|\vec{\eta}| + 2n + \gamma) (\mathbb{1}_{P_{j,\ell}} * |\hat{\kappa}_0|)(U_{j,\ell}^{-\top} \vec{\eta}) d\vec{\eta} \\ |Y_{\lambda,\lambda'}^{**}| &\lesssim 2^{\frac{j-j'}{2}} \int \|\tilde{U}_{j',\ell'}^{j,\ell}\|^{2n} \frac{1}{w(\lambda)w(\lambda')} (\mathbb{1}_{P_{j,\ell}} * |\hat{\kappa}_0|)(U_{j,\ell}^{-\top} \vec{\eta}) (\mathbb{1}_{P_{j',\ell'}} * |\hat{\kappa}_0|)(U_{j,\ell}^{-\top} \vec{\eta}) d\vec{\eta} \end{aligned}$$

Step 4 – Dealing with the convolution terms: Ultimately, we have to find a restriction on j, ℓ in terms of fixed j', ℓ' to be able to sum (5.9) – see Step 6. The way to obtain this restriction is to inspect the supports of $\hat{\psi}_{j,\ell}$ and $\hat{\psi}_{j',\ell'}$ and see where the intersection is non-empty. However, the convolution $\hat{\psi}_{j,\ell} * M_\lambda \hat{\kappa}$ is supported on the Minkowski sum $\text{supp } \hat{\psi}_{j,\ell} + \text{supp } \hat{\kappa}$, and since the latter summand may be unbounded we have to tackle this problem a little differently.

The idea now is to decompose \mathbb{R}^d in such a way, that the increases in non-empty intersections (and other contributions arising therefrom) are offset by the decay of $\hat{\kappa}$. To this end, choose m_0 as the minimal $m \in \mathbb{N}$ such that $2^m \geq r_0$ (from the decay condition (5.2)). Then \mathbb{R}^d may be written as the direct sum

$$\mathbb{R}^d = (P_{j,\ell} + B_{\mathbb{R}^d}(0, 2^{m_0})) \dot{\cup} \bigcup_{m \geq m_0} (P_{j,\ell} + B_{\mathbb{R}^d}(0, 2^{m+1})) \setminus (P_{j,\ell} + B_{\mathbb{R}^d}(0, 2^m)),$$

which we choose to abbreviate by setting $P_{j,\ell}^m := P_{j,\ell} + B_{\mathbb{R}^d}(0, 2^m)$, as well as $Q_{j,\ell}^m := P_{j,\ell}^{m+1} \setminus P_{j,\ell}^m$, thus

$$\mathbb{R}^d = P_{j,\ell}^{m_0} \dot{\cup} \bigcup_{m \geq m_0} P_{j,\ell}^{m+1} \setminus P_{j,\ell}^m = P_{j,\ell}^{m_0} \dot{\cup} \bigcup_{m \geq m_0} Q_{j,\ell}^m.$$

To see the gain of the above decomposition, we consider the convolution terms appearing in the estimates (5.17). In particular, for $\vec{\xi} \in Q_{j,\ell}^m$ and $\vec{\zeta} \in P_{j,\ell}^{m_0}$, the construction of the $Q_{j,\ell}^m$ yields $|\xi - \zeta| \geq 2^m - 2^{m_0}$, which we can exploit to estimate (using $q > d$)

$$(\mathbb{1}_{P_{j,\ell}} * |\hat{\kappa}_0|)(\vec{\xi}) = \int \mathbb{1}_{P_{j,\ell}}(\vec{\zeta}) |\hat{\kappa}_0(\vec{\xi} - \vec{\zeta})| d\vec{\zeta} \leq \int_{|\vec{\zeta}| \geq 2^m - 2^{m_0}} |\hat{\kappa}_0(\vec{\zeta})| d\vec{\zeta} \stackrel{(5.2)}{\leq} \int_{2^m - 2^{m_0}}^{\infty} \frac{c_0}{r^q} r^{d-1} d\Omega dr \lesssim 2^{-m(q-d)}. \quad (5.18)$$

Step 5 – Dealing with the anisotropic terms: The terms $\vec{s} \cdot \vec{\xi}$, respectively $\vec{s} \cdot U_{j,\ell}^{-\top} \eta = U_{j,\ell}^{-1} \vec{s} \cdot \eta$ after the transformation, can get very large ($\sim 2^j$), in particular if \vec{s} is close to $P_{j,\ell}$ – the support of $\hat{\psi}_{j,\ell}$. It is this behaviour that has to be counteracted by the preconditioning, since we would not be able to bound this term otherwise – together with (5.8), (5.4) implies

$$|U_{j,\ell}^{-1} \vec{s}| \leq w(\lambda), \quad \text{and} \quad |U_{j,\ell}^{-1} \vec{s}| \lesssim \max(2^{j-j'}, 1) (w(\lambda') + 2^{j'} \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'})) \lesssim w(\lambda'), \quad (5.19)$$

which conveniently cancels with the weights in the denominator.

Finally, we need the inclusion (5.6) for estimating both $|\vec{\eta}|$ as well as the volume of the integral. This illustrates another property of the transformation we employed, since, in essence (resp. as a consequence of the construction), it transforms the highly anisotropic sets $P_{j,\ell}$ back into a subset of a ball around the origin, see also Figure A.1.

Step 6 – The condition for j and ℓ : As noted above, to be able to sum j and ℓ , we need a restriction for these indices in terms of j', ℓ' . For constant (or compactly supported) $\hat{\kappa}$, such a condition follows naturally from the fact that the integral of the inner product is zero when the supports of the functions $\hat{\psi}_{j,\ell}$ and $\hat{\psi}_{j',\ell'}$ do not intersect. For general $\hat{\kappa}$, we have to consider these intersections for the decomposition we introduced above. Here, the concentric structure of the $Q_{j,\ell}^m$ is irrelevant, it suffices to consider the sets $P_{j,\ell}^{m+1} \supseteq Q_{j,\ell}^m$. The main argument in this respect is (5.8), which basically says that for $j \geq m_{>} + 3$, we are able to restrict j and ℓ in relation to j', ℓ' , whereas for $j \leq m_{>} + 2$, we have to assume the worst-case scenario of all indices contributing to the sum (or rather, a more precise estimate doesn't change anything in this case).

An immediate consequence is, that all terms of the form $2^{j-j'}$ can be estimated

$$2^{j-j'} \leq \begin{cases} 2^{|j-j'|}, & j \geq m_{>} + 3 \\ 2^j, & j \leq m_{>} + 2 \end{cases} \lesssim 2^{m_{>}}. \quad (5.20)$$

Similarly, we can estimate the norm of $\tilde{U}_{j',\ell'}^{j,\ell}$. After appealing to a (5.5), we apply (5.8), first the condition in ℓ for (*) and then in j for (**), and finally (5.20), to arrive at

$$\begin{aligned} \|\tilde{U}_{j',\ell'}^{j,\ell}\| &\stackrel{(5.5)}{\lesssim} \max(2^{j-j'}, 1) + 2^j \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}) \stackrel{(*)}{\leq} \begin{cases} \max(2^{j-j'}, 1) + 5c_\omega 2^{m_{>}+j-j'}, & j \geq m_{>} + 3 \\ \max(2^{j-j'}, 1) + \frac{\pi}{2} 2^j, & j \leq m_{>} + 2 \end{cases} \\ &\lesssim \begin{cases} 2^{m_{>}+|j-j'|}, & j \geq m_{>} + 3 \\ 2^j, & j \leq m_{>} + 2 \end{cases} \stackrel{(**)}{\lesssim} 2^{m_{>}}. \end{aligned} \quad (5.21)$$

Applying the estimates (5.6) and (5.18)–(5.21), we see that (by definition, $m = m' = 0$ for $Y_{\lambda, \lambda'}^{00}$)

$$|Y_{\lambda, \lambda'}^{00}| \lesssim \int_{U_{j, \ell}^\top(P_{j, \ell} \cap P_{j', \ell'})} d\vec{\eta} \quad (5.22)$$

$$|Y_{\lambda, \lambda'}^{0*}| \lesssim \int_{U_{j, \ell}^\top(P_{j, \ell} \cap P_{j', \ell'}^{m_0})} d\vec{\eta} + \sum_{m \geq m_0} 2^{-m(q-d-2n-\frac{1}{2})} \int_{U_{j, \ell}^\top(P_{j, \ell} \cap Q_{j', \ell'}^m)} d\vec{\eta} \quad (5.23)$$

$$|Y_{\lambda, \lambda'}^{*0}| \lesssim \int_{U_{j, \ell}^\top(P_{j, \ell}^{m_0} \cap P_{j', \ell'})} d\vec{\eta} + \sum_{m \geq m_0} 2^{-m(q-d-2n-\frac{3}{2})} \int_{U_{j, \ell}^\top(Q_{j, \ell}^m \cap P_{j', \ell'})} d\vec{\eta} \quad (5.24)$$

$$|Y_{\lambda, \lambda'}^{**}| \lesssim \int_{U_{j, \ell}^\top(P_{j, \ell}^{m_0} \cap P_{j', \ell'}^{m_0})} d\vec{\eta} + \sum_{m \geq m_0} 2^{-m(q-d-2n-\frac{1}{2})} \int_{U_{j, \ell}^\top(P_{j, \ell}^{m_0} \cap Q_{j', \ell'}^m)} d\vec{\eta} + \dots \quad (5.25)$$

$$\dots + \sum_{m \geq m_0} 2^{-m(q-d-2n-\frac{1}{2})} \int_{U_{j, \ell}^\top(Q_{j, \ell}^m \cap P_{j', \ell'}^{m_0})} d\vec{\eta} + \sum_{m, m' \geq m_0} 2^{-(m+m')(q-d-2n-\frac{1}{2})} \int_{U_{j, \ell}^\top(Q_{j, \ell}^m \cap Q_{j', \ell'}^{m'})} d\vec{\eta}$$

Step 7 – Summing \vec{k} : Thus far, we have omitted the case $\vec{k} = U_{j', \ell'}^{j, \ell} \vec{k}'$ – in fact, to sum over \vec{k} , we need treat even more elements differently. In order to estimate the term $|\vec{k} - U_{j', \ell'}^{j, \ell} \vec{k}'|$, we choose $K_{j', \ell'}^{j, \ell} \vec{k}' \in \mathbb{Z}^d$ as a (possibly non-unique) closest lattice element to $U_{j', \ell'}^{j, \ell} \vec{k}'$ (for example by rounding every component to the nearest integer), which may be interpreted as a projection of $U_{j', \ell'}^{j, \ell} \vec{k}'$ onto the lattice \mathbb{Z}^d . Then $|K_{j', \ell'}^{j, \ell} \vec{k}' - U_{j', \ell'}^{j, \ell} \vec{k}'| \leq \frac{\sqrt{d}}{2}$, and if we restrict $\vec{k} \in \mathbb{Z}^d$ such that $|\vec{k} - K_{j', \ell'}^{j, \ell} \vec{k}'| \geq \sqrt{d}$, it holds that

$$|\vec{k} - U_{j', \ell'}^{j, \ell} \vec{k}'| \geq |\vec{k} - K_{j', \ell'}^{j, \ell} \vec{k}'| - \frac{\sqrt{d}}{2} \geq \frac{1}{2} |\vec{k} - K_{j', \ell'}^{j, \ell} \vec{k}'|. \quad (5.26)$$

For $\vec{k} \in \mathbb{Z}^d$ such that $|\vec{k} - K_{j', \ell'}^{j, \ell} \vec{k}'| < \sqrt{d}$, we retrace the derivation of all above estimates without the partial integration, which, in effect, only eliminates the divisor $|\vec{k} - U_{j', \ell'}^{j, \ell} \vec{k}'|^{2n}$ (and reduces the constants). Putting the estimates for (5.22)–(5.25) together, we arrive at

$$\mathbf{F}_{\lambda, \lambda'} \lesssim |\vec{k} - U_{j', \ell'}^{j, \ell} \vec{k}'|^{-2n} (|Y_{\lambda, \lambda'}^{00}| + |Y_{\lambda, \lambda'}^{0*}| + |Y_{\lambda, \lambda'}^{*0}| + |Y_{\lambda, \lambda'}^{**}|) =: Z_{\lambda, \lambda'}^{00} + Z_{\lambda, \lambda'}^{0*} + Z_{\lambda, \lambda'}^{*0} + Z_{\lambda, \lambda'}^{**},$$

for $|\vec{k} - K_{j', \ell'}^{j, \ell} \vec{k}'| \geq \sqrt{d}$, and similarly for $|\vec{k} - K_{j', \ell'}^{j, \ell} \vec{k}'| < \sqrt{d}$,

$$\mathbf{F}_{\lambda, \lambda'} \lesssim |Y_{\lambda, \lambda'}^{00}| + |Y_{\lambda, \lambda'}^{0*}| + |Y_{\lambda, \lambda'}^{*0}| + |Y_{\lambda, \lambda'}^{**}| =: Z_{\lambda, \lambda'}^{00} + Z_{\lambda, \lambda'}^{0*} + Z_{\lambda, \lambda'}^{*0} + Z_{\lambda, \lambda'}^{**}.$$

Note that the different cases for \vec{k} are incorporated in the definition of the Z -terms.

The intention now is to prove (5.9) by showing

$$\begin{aligned} \sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} |\mathbf{F}_{\lambda, \lambda'}|^p &\lesssim \sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} (Z_{\lambda, \lambda'}^{00} + Z_{\lambda, \lambda'}^{0*} + Z_{\lambda, \lambda'}^{*0} + Z_{\lambda, \lambda'}^{**})^p \\ &\leq \sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} (Z_{\lambda, \lambda'}^{00})^p + (Z_{\lambda, \lambda'}^{0*})^p + (Z_{\lambda, \lambda'}^{*0})^p + (Z_{\lambda, \lambda'}^{**})^p < \infty, \end{aligned} \quad (5.27)$$

where the second inequality requires $p \leq 1$. All four terms have the same structure in \vec{k} – namely, the integrals do not depend on this parameter. Thus we may calculate the sum over \vec{k} separately, which crucially requires the condition $p > \frac{d}{2n}$,

$$\sum_{\substack{\vec{k} \in \mathbb{Z}^d \\ |\vec{k} - K_{j', \ell'}^{j, \ell} \vec{k}'| \geq \sqrt{d}}} \frac{1}{|\vec{k} - U_{j', \ell'}^{j, \ell} \vec{k}'|^{2np}} \stackrel{(5.26)}{\leq} \sum_{\substack{\vec{k} \in \mathbb{Z}^d \\ |\vec{k} - K_{j', \ell'}^{j, \ell} \vec{k}'| \geq \sqrt{d}}} \frac{2^{2np}}{|\vec{k} - K_{j', \ell'}^{j, \ell} \vec{k}'|^{2np}} = \sum_{\substack{\vec{k} \in \mathbb{Z}^d \\ |\vec{k}| \geq \sqrt{d}}} \frac{2^{2np}}{|\vec{k}|^{2np}} =: G_{d, 2np} < \infty.$$

The remaining sum over $\vec{k} \in \mathbb{Z}^d$: $|\vec{k} - K_{j',\ell'}^{j,\ell} \vec{k}'| \leq \sqrt{d}$ has at most $\mathcal{O}(\sqrt{d}^d)$ terms. Taken together, this implies (recall that the Y -terms do not depend on \vec{k} and \vec{k}')

$$\sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda} |\mathbf{F}_{\lambda,\lambda'}|^p \lesssim \sum_{j \in \mathbb{N}_0} \sum_{\ell=0}^{L_j} |Y_{\lambda,\lambda'}^{00}|^p + |Y_{\lambda,\lambda'}^{0*}|^p + |Y_{\lambda,\lambda'}^{*0}|^p + |Y_{\lambda,\lambda'}^{**}|^p.$$

Step 8 – Estimating the Number of Intersections:

As the last important tool to show the finiteness of (5.27), we show one more estimate – the number of non-empty intersections $P_{j,\ell}^m \cap P_{j',\ell'}^{m'}$ in terms of ℓ . Recalling $\alpha_j = 2^{-j+1}$, we derive that (for $j, j' \geq m_{>} + 1$)

$$\begin{aligned} N_{j,j',\ell'}^{m,m'} &:= \#\{\ell \in \{0, \dots, L_j\} : P_{j,\ell}^m \cap P_{j',\ell'}^{m'} \neq \emptyset\} \leq \#\{\ell \in \{0, \dots, L_j\} : \mathcal{P}_{\mathbb{S}^{d-1}}(P_{j,\ell}^m) \cap \mathcal{P}_{\mathbb{S}^{d-1}}(P_{j',\ell'}^{m'}) \neq \emptyset\} \\ &= \#\{\ell \in \{0, \dots, L_j\} : B_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \alpha_j^m) \cap B_{\mathbb{S}^{d-1}}(\vec{s}_{j',\ell'}, \alpha_{j'}^{m'}) \neq \emptyset\} \\ &\stackrel{(5.7)}{\leq} \#\{\ell \in \{0, \dots, L_j\} : B_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, c_\omega 2^{m-j}) \cap B_{\mathbb{S}^{d-1}}(\vec{s}_{j',\ell'}, c_\omega 2^{m'-j'}) \neq \emptyset\} \\ &\stackrel{(A.2)}{\leq} \frac{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}_{j',\ell'}, 3c_\omega 2^{m_{>} - j_{<}}))}{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \frac{1}{3} 2^{-j}))} \stackrel{(A.1)}{\leq} \frac{C_d}{c_d} (9c_\omega 2^{j-j_{<}+m_{>}})^{d-1} \lesssim 2^{(|j-j'|+m_{>})(d-1)}. \end{aligned} \quad (5.28)$$

In particular, the estimate is independent of the choice of ℓ' .

Step 9 – Summing j and ℓ : The following procedure is very similar for all four terms (5.22)–(5.25), we demonstrate the procedure with the most difficult term. With $L_j \lesssim 2^{j(d-1)}$ and $\max(m+1, m'+1) = m_{>} + 1$ for (5.28), we have

$$\begin{aligned} \sum_{\lambda \in \Lambda} (Z_{\lambda,\lambda'}^{**})^p &\leq \sum_{j \in \mathbb{N}_0} \sum_{\ell=0}^{L_j} \left(\sum_{m,m' \geq m_0} 2^{-(m+m')(q-d-2n-\frac{1}{2})} \int_{U_{j,\ell}^\top(Q_{j,\ell}^m \cap Q_{j',\ell'}^{m'})} d\vec{\eta} \right)^p \\ &\stackrel{p \leq 1}{\lesssim} \sum_{m,m' \geq m_0} 2^{-p(m+m')(q-d-2n-\frac{1}{2})} \sum_{j \in \mathbb{N}_0} \sum_{\ell=0}^{L_j} \left(\int_{U_{j,\ell}^\top(Q_{j,\ell}^m \cap Q_{j',\ell'}^{m'})} d\vec{\eta} \right)^p \\ &\stackrel{(5.8)}{\leq} \sum_{m,m' \geq m_0} 2^{-p(m+m')(q-d-2n-\frac{1}{2})} \left[\sum_{j=0}^{m_{>}+3} (L_j + 1) \sup_{\ell=0}^{L_j} \left(\int_{U_{j,\ell}^\top(Q_{j,\ell}^m \cap Q_{j',\ell'}^{m'})} d\vec{\eta} \right)^p + \dots \right. \\ &\quad \left. \dots + \sum_{\substack{j \geq m_{>}+4 \\ |j-j'| \leq 2}} N_{j,j',\ell'}^{m+1,m'+1} \sup_{\ell=0}^{L_j} \left(\int_{U_{j,\ell}^\top(Q_{j,\ell}^m \cap Q_{j',\ell'}^{m'})} d\vec{\eta} \right)^p \right] \\ &\stackrel{(5.6)}{\lesssim} \sum_{m,m' \geq m_0} 2^{-p(m+m')(q-d-2n-\frac{1}{2})} \left[\sum_{j=0}^{m_{>}+3} (L_j + 1) 2^{mdp} + \sum_{\substack{j \geq m_{>}+4 \\ |j-j'| \leq 2}} N_{j,j',\ell'}^{m+1,m'+1} 2^{mdp} \right] \\ &\stackrel{(5.28)}{\lesssim} \sum_{m,m' \geq m_0} 2^{-p(m+m')(q-d-2n-\frac{1}{2})} \left[\sum_{j=0}^{m_{>}+3} 2^{j(d-1)+m_{>}dp} + \sum_{\substack{j \geq m_{>}+4 \\ |j-j'| \leq 2}} 2^{(|j-j'|+m_{>}+1)(d-1)+m_{>}dp} \right] \\ &\lesssim \sum_{m,m' \geq m_0} 2^{-p(m+m')(q-d-2n-\frac{1}{2})} 2^{m_{>}(dp+d-1)} \leq \sum_{m,m' \geq m_0} 2^{-p(m+m')(q-2d-2n-\frac{1}{2}-\frac{d-1}{p})} \\ &= \sum_{m \geq m_0} 2^{-mp(q-2d-2n-\frac{1}{2}-\frac{d-1}{p})} \sum_{m' \geq m_0} 2^{-m'p(q-2d-2n-\frac{1}{2}-\frac{d-1}{p})} = \left(\frac{c^{m_0}}{1-c} \right)^2 < \infty. \end{aligned}$$

Here, $c := 2^{-p(q-2d-2n-\frac{1}{2}-\frac{d-1}{p})}$ can be summed geometrically, since $q > 2d + 2n + \frac{3}{2} + \frac{d-1}{p}$ implies $c < 1$.

As we have now estimated every term in (5.27) independently of λ' , taking the supremum does not change anything and the proof is finished. \square

5.3 Sparsity of \mathbf{P}

Recall that in order for Algorithm 3.6 to converge in optimal complexity (compare Theorem 3.11), not only \mathbf{F} but also \mathbf{P} need to be compressible (resp. sparse). This is formulated in the following theorem.

Theorem 5.9. *Again, let $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda}$ satisfy Assumption 4.7 for $2n$ with $\frac{d}{2} < n \in \mathbb{N}$, and choose $p \in \mathbb{R}$ such that $1 > p > \frac{d}{2n}$. Then the projection \mathbf{P} , see (4.5), is p -sparse in this frame – in other words,*

$$\|\mathbf{P}\|_{\ell^p(\Lambda) \rightarrow \ell^p(\Lambda)} = \left\| \mathbf{W} \langle \Phi, \Phi \rangle_{L^2} \mathbf{W}^{-1} \right\|_{\ell^p(\Lambda) \rightarrow \ell^p(\Lambda)} < \infty.$$

As the proof proceeds exactly along the lines of Theorem 5.4 (but with substantial simplifications due to the lack of the operator A), we leave it to the reader.

6 Main results

The results so far allow us to formulate the following corollary to Theorem 4.3, which, in essence, states that the complexity of **modSOLVE** is *linear* with respect to the number of relevant coefficients of the discretisation.

Corollary 6.1. *Assume that \mathbf{f} is σ^* -optimal (compare Definition 3.10) and that the system $\mathbf{F}\mathbf{u} = \mathbf{f}$ has a solution $\mathbf{u} \in \ell_w^p(\Lambda)$ for $\sigma \in (0, \sigma^*)$ and $p := \frac{1}{\frac{1}{2} + \sigma}$. Then the solution $\mathbf{u}_\varepsilon := \mathbf{modSOLVE}[\varepsilon, \mathbf{F}, \mathbf{P}, \mathbf{f}]$ of the ridgelet-based solver recovers this approximation rate – i.e.*

$$\#\text{supp } \mathbf{u}_\varepsilon \lesssim \varepsilon^{-1/\sigma} |\mathbf{u}|_{\ell_w^p(\Lambda)}^{1/\sigma},$$

and the number of arithmetic operations is at most a multiple of $\varepsilon^{-1/\sigma} |\mathbf{u}|_{\ell_w^p(\Lambda)}^{1/\sigma}$.

Finally, the last assumption – that the discretisation of typical solutions are in $\ell_w^p(\Lambda)$ – is also satisfied by the ridgelet discretisation. The proof of this theorem is based on arguments of [Can01] and is the subject of an upcoming paper [GO14].

Theorem 6.2. *For a function $u \in L^2(\mathbb{R}^d)$ such that $u, \vec{s} \cdot \nabla u \in H^t(\mathbb{R}^d)$ apart from discontinuities across hyperplanes containing \vec{s} . Then $\mathbf{W} \langle \Phi, u \rangle_{L^2} \in \ell_w^p$, the weak ℓ^p -space with $\frac{1}{p} = \frac{t}{d} + \frac{1}{2}$. This is the best possible approximation rate for functions in $H^t(\mathbb{R}^d)$ (even without singularities!).*

To conclude the theoretical discussion, the bottom line is that the presented construction “sparsifies” both the system matrix as well as typical solutions of transport problems (in the sense of compressibility and N -term approximations, respectively), which makes it the ideal candidate for the development of fast algorithms, as underscored also by the results of Corollary 6.1.

7 Numerical Experiments

To underpin the theoretical claims of the paper, we implemented Algorithm 3.5 in MATLAB. We need to stress, however, that as things stand, this is not a competitive solver, but rather a proof-of-concept. There are two main caveats: **APPLY** is not fully adaptive – but rather uses a heuristic based on the distance of the translations – and the necessary quadrature effort is substantial. To a degree, this is the price for sticking close to the theory – another paper describing an implementation based on **FFT** is forthcoming, see [EGO14].

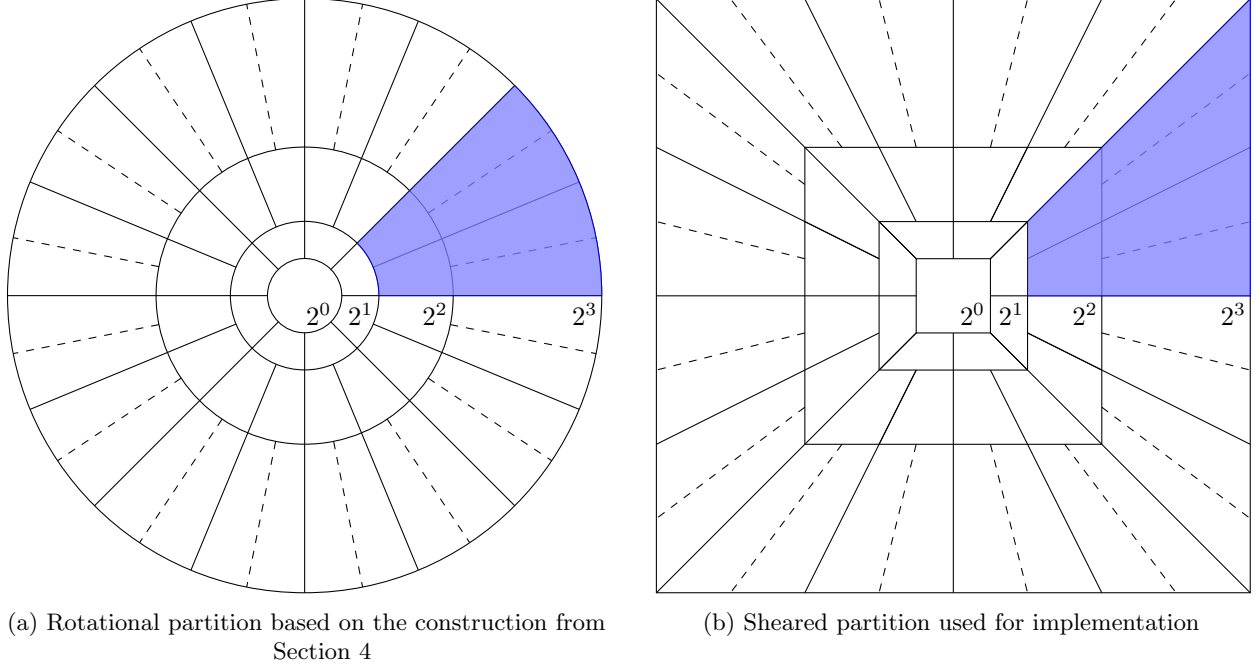


Figure 7.1: Two different partitions in Fourier space, with the support of one ridgelet on scale $j = 2$ shaded in blue¹. The dashed lines are the continuation of the pattern but only become relevant for the following scale $j = 3$.

7.1 Implementation

As a proof-of-concept, the implementation is only 2-dimensional. The first important difference between theory and implementation is that instead of using the rotations $R_{j,\ell}$ to generate the different ridgelets, we use a shearing matrix

$$S_{j,\ell} := \begin{pmatrix} 1 & \frac{\ell}{2^j} \\ 0 & 1 \end{pmatrix}.$$

The advantage of this is that the transformed grid of translations ($U_{j,\ell}\mathbb{Z}^d$, compare Definition 4.2) becomes invariant to the relevant shears and one can avoid tedious interpolation that would be necessary for a faithful implementation based on rotations. The difference between the different partitions of unity is illustrated in Figure 7.1. This change does not affect the theoretical properties, as can be shown using [GKKS14].

As in Section 4, the ridgelets are constructed in the Fourier domain, for a sufficiently smooth transition function between zero and one. In our case, we used

$$t(\xi) = \frac{\exp\left(-\frac{1}{\xi^\alpha}\right)}{\exp\left(-\frac{1}{\xi^\alpha}\right) + \exp\left(-\frac{1}{(1-\xi)^\alpha}\right)}$$

with $\alpha = 1.1$, as opposed to the “classical” choice of $\alpha = 2$ (for an example how $V^{(j,\ell)}$ and W can be constructed from t , see Section B). The reason for this is that the transition for $\alpha = 2$ – while being \mathcal{C}^∞ – nevertheless has a very sharp step around $\xi = 0.5$, which results in an unfavourable localisation in physical space. The choice of $\alpha = 1.1$ alleviates this problem to a degree.

¹For the implementation, the support is actually taken together with its point-symmetric mirror image (around the origin) to reduce the number of parameters.

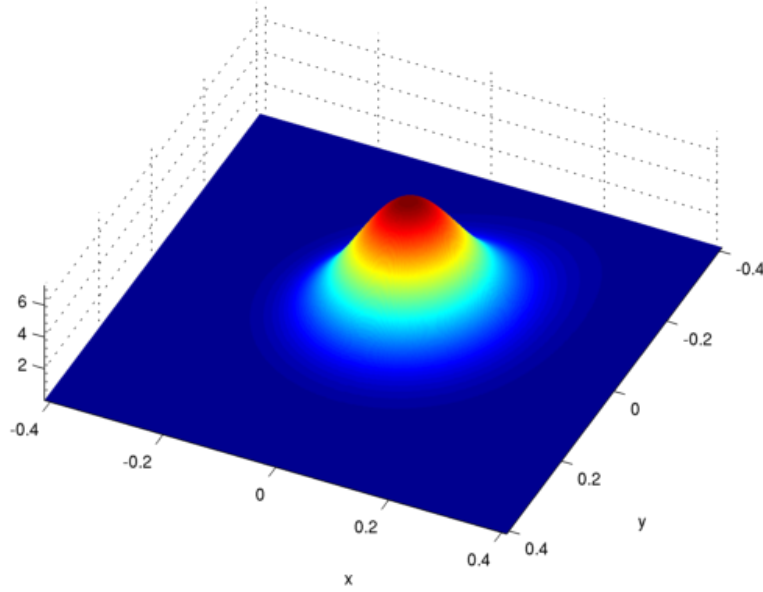


Figure 7.2: Solution to the transport equation for a smooth right-hand side (Gaussian)

We note that – while we have neglected quadrature effort in the theoretical discussion (as is usually the case, see [Ste04]) – the actual computational cost is not negligible. Furthermore, for a good localisation of the solution in physical space, we have chosen a relatively high value for the absorption coefficient (which is constant in the model problem for the implementation). However, all of these caveats can be alleviated by the above-mentioned FFT-based solver in [EGO14], although at the “cost” of a less direct relationship to the theoretical results.

7.2 Convergence of the solver

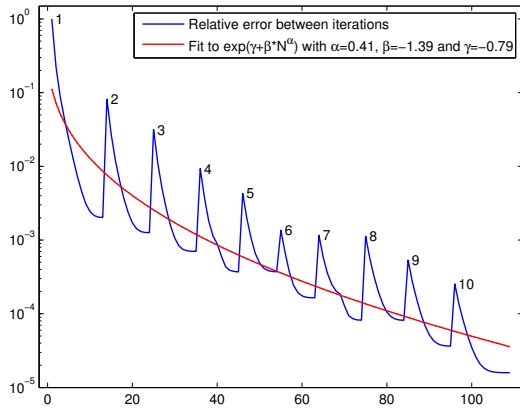
For both smooth and singular functions, we have used Algorithm 3.6 to solve the transport equation. In particular, we have observed that *without* the projection \mathbf{P} , the results deteriorate and eventually diverge, while with the projection, convergence can be observed. To the best of our knowledge, this is the first time where the positive effect of such a projection is observed in practice. It is worth noting however, that the kernels of \mathbf{F} and \mathbf{P} in their respective restrictions to a finite set of coefficients don’t match anymore. This is also observed in the sense that for frames that are “too small”, the projection does not improve the convergence.

7.2.1 Smooth functions

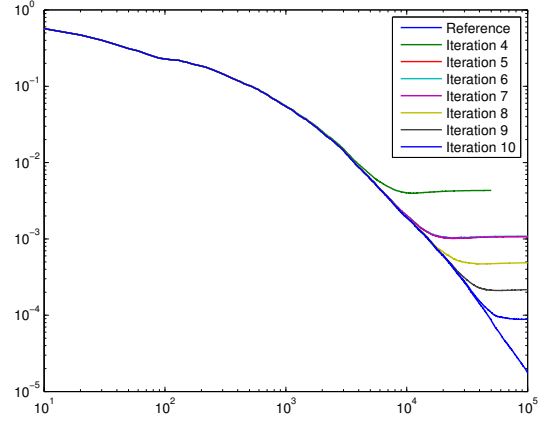
For smooth functions, the algorithm works as expected, as illustrated in Figures 7.2 and 7.3.

7.2.2 Functions with Singularities

The main thrust of the construction, however, is that singularities are resolved with the same N -term approximation (and complexity) as if they weren’t there. We illustrate this with the following example in Figure 7.4 – the right-hand side is a product of a Gaussian times a box function, rotated in a direction that isn’t aligned with any $\vec{s}_{j,\ell}$ in the ridgelet frame (rather arbitrarily constructed from irrational numbers such that it lies in the second octant) – $\vec{s} = \left(\frac{\sqrt{2}}{2}, \frac{\pi}{3}\right)^\top$, normed to 1. We use the same \vec{s} as the transport direction



(a) Relative error between iterations



(b) N -term approximations of outer iterations

Figure 7.3: Convergence of solver to solution in Figure 7.2. Subplot (a) shows the relative error between iterations, while (b) shows the N -term approximations at the end of an outer iteration (beginnings marked in (a)) against the reference solution.

in the differential equation, which means that there is no smoothing orthogonal to \vec{s} , and in particular, the singularities of the right-hand side remain in the solution.

In Figure 7.5, the N -term approximation of the reference solution (which can be calculated explicitly thanks to constant κ), compared against the N -term approximation of the numerical solution after several outer iteration of Algorithm 3.5. Even though the functions have singularities, the N -term approximation can be seen to converge exponentially!

In Figure 7.6, we illustrate the localisation properties of the ridgelet frame – namely, for the singular solution from Figure 7.4, we consider the translations corresponding to the 10000 largest coefficients (of the discretisation up to scale $j = 10$). As can be seen in Figure 7.6, (b)–(l), the higher the scale, the less coefficients are active (also with diminishing maximal Euclidian norms), and the more they are aligned with the location of the singularities. Note that the grid is only refined in one direction, which explains the constant distance in the y -direction. Conversely, in the x -direction, what may appear like a single point are often many points very close to each other.

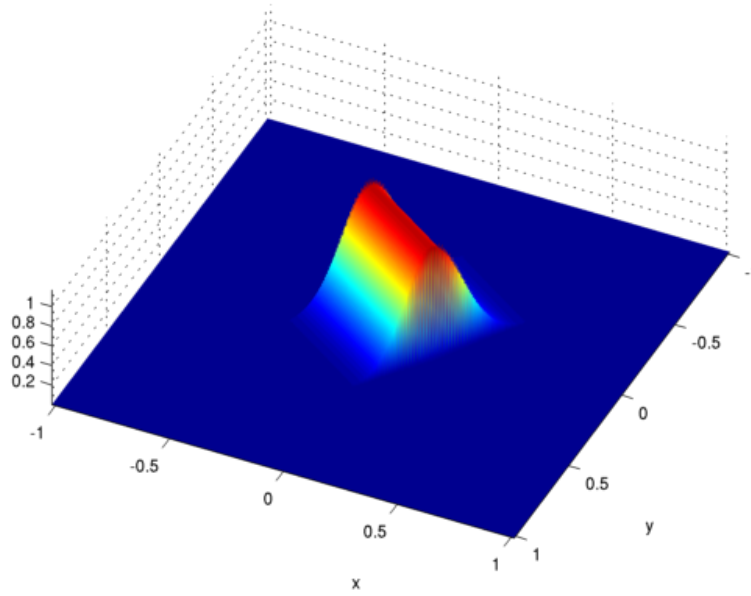


Figure 7.4: Solution to the transport equation for a singular right-hand side (box times Gaussian) with transport direction parallel to the orientation of the singularity.

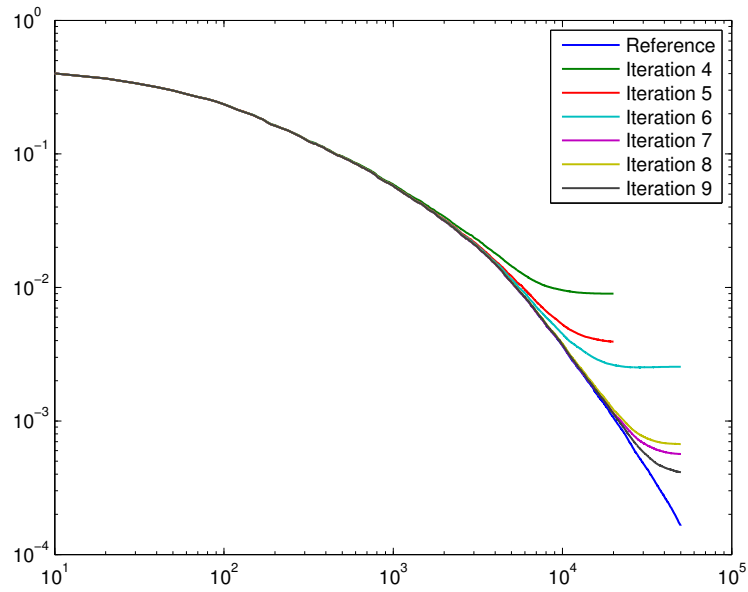


Figure 7.5: Comparison of the N -term approximations of the reference solution, against the N -term approximations of the output of Algorithm 3.5. Even though the functions are singular, the N -term approximations converge exponentially!

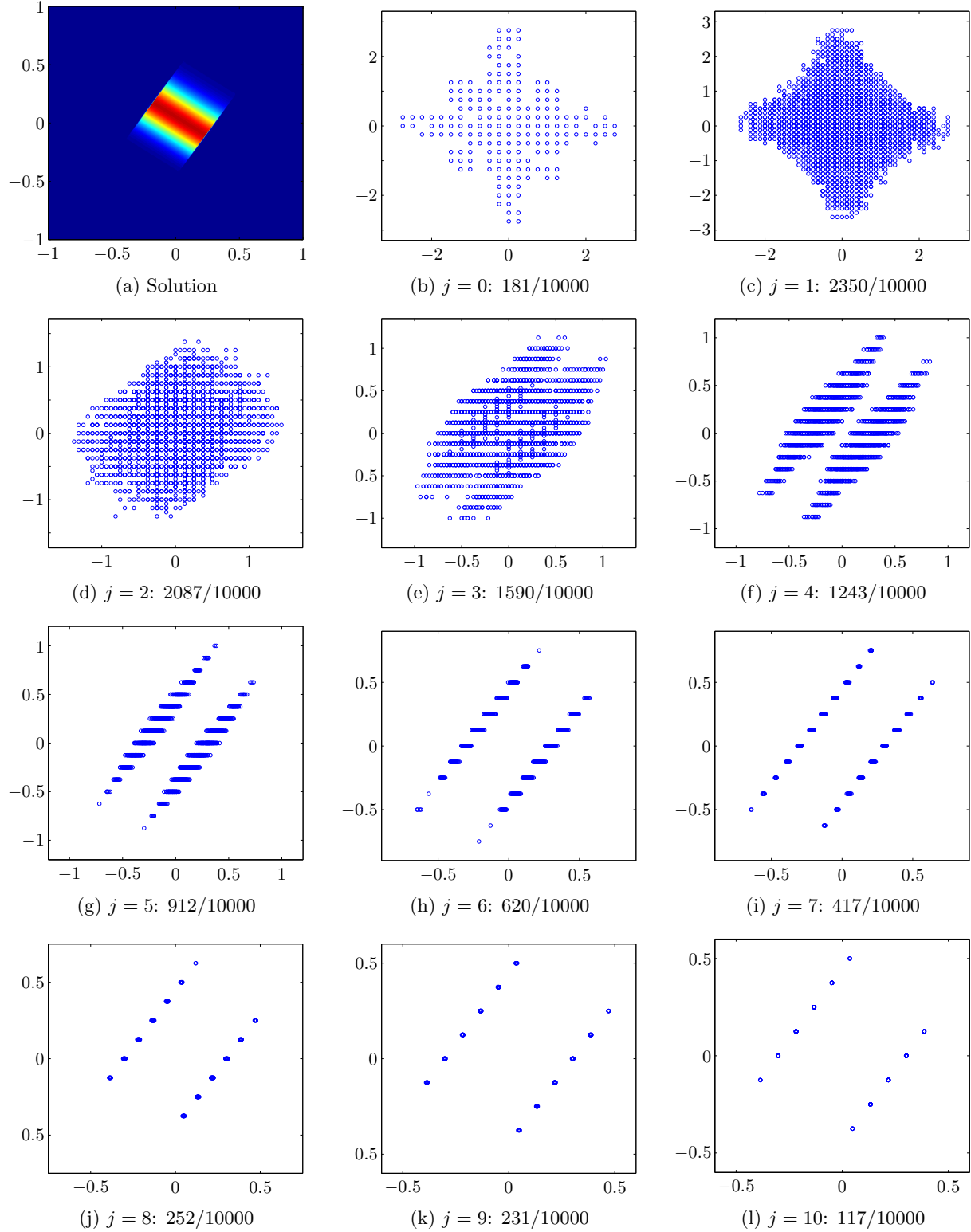


Figure 7.6: Localisation of solution in (a) in the ridgelet frame: Subplots (b)–(l) show the translations corresponding to the 10000 largest coefficients (up to scale $j = 10$) within a given scale. At high scales, only coefficients close to the singularities are active – as expected.

Appendix A Geometric Considerations

A.1 Basic Properties of the Hypersphere

For various estimates, we need properties of the $(d-1)$ -dimensional hypersphere $\mathbb{S}^{d-1} = \{\vec{x} \in \mathbb{R}^d : |\vec{x}| = 1\}$, which we equip it with the geodesic metric

$$\text{dist}_{\mathbb{S}^{d-1}}(\vec{s}, \vec{s}') = \arccos(\vec{s} \cdot \vec{s}').$$

Naturally, an equivalent metric would make no difference other than changing some constants. For convenience we extend $\text{dist}_{\mathbb{S}^{d-1}}$ to a pseudo-metric on \mathbb{R}^d by taking

$$\text{dist}_{\mathbb{S}^{d-1}}(\vec{x}, \vec{x}') = \arccos(\mathcal{P}_{\mathbb{S}^{d-1}}(\vec{x}) \cdot \mathcal{P}_{\mathbb{S}^{d-1}}(\vec{x}')) = \arccos\left(\frac{\vec{x}}{|\vec{x}|} \cdot \frac{\vec{x}'}{|\vec{x}'|}\right).$$

Remark A.1. For $\vec{s} \in \mathbb{S}^{d-1}$, straight-forward calculus shows that the geodesic metric is equivalent to the Euclidian metric,

$$|\vec{s} - \vec{s}'| \leq \arccos(\vec{s} \cdot \vec{s}') \leq \frac{\pi}{2} |\vec{s} - \vec{s}'|.$$

The construction of the ridgelet frame uses window functions supported on “balls” in this metric space,

$$B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha) := \{\vec{s}' \in \mathbb{S}^{d-1} : \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}, \vec{s}') < \alpha\},$$

appropriately called *hyperspherical caps*. For $\alpha > \pi$, we define $B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha)$ as the whole sphere \mathbb{S}^{d-1} . These are closely related with the solid angle corresponding to α , which we estimate in the following lemma.

Lemma A.2. *The d -dimensional solid angle corresponding to opening angle α can be estimated by*

$$\Omega_d(\alpha) := \frac{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha_2))}{\mu(\mathbb{S}^{d-1})} \lesssim \alpha^{d-1},$$

where μ is the canonical surface measure of \mathbb{S}^{d-1} . For $\alpha_1 \leq \frac{\pi}{2}$ and arbitrary $\alpha_2 > 0$,

$$\frac{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha_2))}{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha_1))} \leq \frac{C_d}{c_d} \left(\frac{\alpha_2}{\alpha_1}\right)^{d-1}, \quad (\text{A.1})$$

where c_d, C_d are constants that only depend on d .

Proof. The area of the hyperspherical cap for arbitrary but fixed $\vec{s} \in \mathbb{S}^{d-1}$ and opening angle $\alpha \in [0, \frac{\pi}{2}]$ can be calculated (see [Li11]) as follows. The idea is that the intersection of \mathbb{S}^{d-1} with an affine hyperplane perpendicular to \vec{s} is a $(d-2)$ -dimensional sphere (if the intersection is not empty) – all its points have the same angle to \vec{s} , say ϑ , in which case the radius of this lower-dimensional sphere is $\sin \vartheta$. Integrating over this angle ϑ between 0 and α will then yield the desired area. From this we obtain the solid angle by dividing through the area of the whole sphere. Denoting by

$$S_i(r) = \frac{2\pi^{\frac{i+1}{2}}}{\Gamma(\frac{i+1}{2})} r^i$$

the surface area of the i -dimensional hypersphere with radius r , we calculate

$$\begin{aligned} \Omega_d(\alpha) &= \frac{1}{S_{d-1}(1)} \mu(B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha)) = \frac{1}{S_{d-1}(1)} \int_0^\alpha S_{d-2}(\sin \vartheta) d\vartheta \\ &= \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int_0^\alpha \underbrace{(\sin \vartheta)^{d-2}}_{\leq \vartheta} d\vartheta \leq \frac{1}{B(\frac{d-1}{2}, \frac{1}{2})} \frac{1}{d-1} \alpha^{d-1} \lesssim \alpha^{d-1}, \end{aligned}$$

where $B(x, y)$ is the beta function. The same argument can be used to yield (use e.g. $\sin \vartheta \geq \frac{\vartheta}{2}$ for the lower estimate), for $\alpha \in [0, \frac{\pi}{2}]$,

$$c_d \alpha^{d-1} \leq \mu(B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha)) \leq C_d \alpha^{d-1}.$$

Subsequently, for two angles α_1, α_2 , the inequality

$$\mu(B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha_2)) \leq C_d \alpha_2^{d-1} \leq \frac{C_d}{c_d} \left(\frac{\alpha_2}{\alpha_1} \right)^{d-1} \mu(B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha_1))$$

holds as long as $\alpha_1 \leq \frac{\pi}{2}$, which finishes the proof. \square

A.2 Construction of the $\vec{s}_{j,\ell}$

The construction of the $\psi_{j,\ell}$ (see [Gro11]) requires a sequence of points on the sphere with particular properties. The following proposition collects these properties and some consequences.

Proposition A.3. *For fixed $\alpha > 0$, there exists a sequence $\{\vec{s}_\ell\}_{\ell \in \{0, \dots, L\}}$ such that*

$$\bigcup_{\ell=0}^L B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \alpha) = \mathbb{S}^{d-1}, \quad B_{\mathbb{S}^{d-1}}\left(\vec{s}_\ell, \frac{\alpha}{3}\right) \text{ are pairwise disjoint,}$$

and $L \lesssim \left(\frac{1}{\alpha}\right)^{d-1}$. Additionally, for an arbitrary cap of opening angle α' (and possibly using dilation $q, q' > 0$), the number of non-empty intersections of the sequence with this cap can be estimated by

$$\#\left\{\ell \in \{0, \dots, L\} : B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, q\alpha) \cap B_{\mathbb{S}^{d-1}}(\vec{s}', q'\alpha')\right\} \leq \frac{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}', 3(q\alpha)_>))}{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \frac{\alpha}{3}))}, \quad (\text{A.2})$$

where $(q\alpha)_> := \max(q\alpha, q'\alpha')$.

Proof. The construction of the sequence (and the idea for the estimate below) can be found in [BN07]. We note that for $\alpha > \pi$, we simply choose $\vec{s}_0 := \vec{e}_1$.

To estimate the number of intersections for two such sequences, we let $\nu(\vec{s}', \alpha, \alpha') := \{\vec{s} \in \mathbb{S}^{d-1} : B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha) \cap B_{\mathbb{S}^{d-1}}(\vec{s}', \alpha') \neq \emptyset\}$ and see that

$$\bigcup_{\vec{s} \in \nu(\vec{s}', \alpha, \alpha')} B_{\mathbb{S}^{d-1}}(\vec{s}, \alpha) \subseteq B_{\mathbb{S}^{d-1}}(\vec{s}', \alpha' + 2\alpha) \subseteq B_{\mathbb{S}^{d-1}}(\vec{s}', 3\alpha_>).$$

In particular, all member sets of our covering having non-empty intersection with $B_{\mathbb{S}^{d-1}}(\vec{s}', \alpha')$ are contained in $B_{\mathbb{S}^{d-1}}(\vec{s}', 3\alpha_>)$. Consequently, the number of non-empty intersections $B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \alpha) \cap B_{\mathbb{S}^{d-1}}(\vec{s}', \alpha')$ can be estimated by assuming that $B_{\mathbb{S}^{d-1}}(\vec{s}', 3\alpha_>)$ is perfectly filled out by the disjoint sets $B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \frac{\alpha}{3})$. In other words,

$$\#\left\{\ell \in \{0, \dots, L\} : B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \alpha) \cap B_{\mathbb{S}^{d-1}}(\vec{s}', \alpha')\right\} \leq \frac{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}', 3\alpha_>))}{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \frac{\alpha}{3}))}.$$

In particular, by setting $\alpha' > \frac{\pi}{3}$, we obtain

$$\#\{0, \dots, L\} \leq \frac{\mu(\mathbb{S}^{d-1})}{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \frac{\alpha}{3}))} \stackrel{(\text{A.1})}{\lesssim} \frac{1}{\alpha^{d-1}},$$

which is the desired estimate. In the case that dilations $q, q' > 0$ are applied after the construction, we argue in a similar fashion, now using the disjoint sets $B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \frac{\alpha}{3})$ to fill out the dilated sets $B_{\mathbb{S}^{d-1}}(\vec{s}', 3(q\alpha)_>) \supseteq B_{\mathbb{S}^{d-1}}(\vec{s}', q\alpha + 2q'\alpha')$, thus

$$\#\left\{\ell \in \{0, \dots, L\} : B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, q\alpha) \cap B_{\mathbb{S}^{d-1}}(\vec{s}', q'\alpha')\right\} \leq \frac{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}', 3q_>\alpha_>))}{\mu(B_{\mathbb{S}^{d-1}}(\vec{s}_\ell, \frac{\alpha}{3}))}.$$

This finishes the proof. \square

A.3 Properties of $U_{j,\ell}$ and $P_{j,\ell}$

Lemma A.4. For $\vec{s}, \vec{s}' \in \mathbb{S}^{d-1}$ there exist rotations $R_{\vec{s}}, R_{\vec{s}'}$ which map \vec{e}_1 to \vec{s}, \vec{s}' respectively, such that

$$\|R_{\vec{s}} - R_{\vec{s}'}\| \lesssim \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}, \vec{s}'). \quad (\text{A.3})$$

Proof. We only consider the case $d > 2$ since the other cases are trivial.

Step 1: For any fixed $\delta > 0$ it suffices to consider points with

$$\text{dist}_{\mathbb{S}^{d-1}}(\vec{s}, \vec{s}') < \delta. \quad (\text{A.4})$$

To see this, assume that

$$\text{dist}_{\mathbb{S}^{d-1}}(\vec{s}, \vec{s}') \geq \delta.$$

In this case we have the trivial estimate

$$\|R_{\vec{s}} - R_{\vec{s}'}\| \leq \|R_{\vec{s}}\| + \|R_{\vec{s}'}\| = 2 \leq \frac{2}{\delta} \delta \leq \frac{2}{\delta} \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}, \vec{s}').$$

Therefore it is no loss in generality to assume that (A.4) holds for a suitably small and henceforth fixed $\delta > 0$. This implies, by applying a suitable rotation, that both \vec{s}, \vec{s}' can be assumed to lie in a fixed small neighborhood around \vec{e}_1 .

Step 2: We first consider the case $d = 3$. Then we can write each $\vec{t} \in \mathbb{S}^{d-1}$ as

$$\vec{t} = (\cos(\theta) \cos(\psi), -\cos(\theta) \sin(\psi), \sin(\theta))^\top.$$

For \vec{t} in a sufficiently small neighborhood of \vec{e}_1 , the assignment

$$\Phi: \vec{t} \mapsto (\theta, \psi)$$

is smooth. Define

$$R_{\vec{t}} := \begin{pmatrix} \cos(\theta) \cos(\psi) & \sin(\psi) & -\sin(\theta) \cos(\psi) \\ -\cos(\theta) \sin(\psi) & \cos(\psi) & \sin(\theta) \sin(\psi) \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix},$$

where (θ, ψ) are defined by $\Phi(\vec{t})$. Since Φ is smooth, the assignment $\vec{t} \mapsto R_{\vec{t}}$ is smooth in a neighborhood of \vec{e}_1 and therefore Lipschitz – this implies (A.3) for $d = 3$.

Step 3: Now assume d general. Pick an ONB $\{\vec{e}_1, \vec{e}_2', \dots, \vec{e}_d'\}$ such that \vec{s}, \vec{s}' lie in the span of $\vec{e}_1, \vec{e}_2', \vec{e}_3'$ and set

$$\mathcal{E}_{\vec{s}, \vec{s}'} = (\vec{e}_1 | \vec{e}_2' | \dots | \vec{e}_d').$$

Using this coordinate system, we may define the matrix $R_{\vec{t}}$ for any $\vec{t} = v_1(\vec{t})\vec{e}_1 + v_2(\vec{t})\vec{e}_2' + v_3(\vec{t})\vec{e}_3' \in \text{span}\{\vec{e}_1, \vec{e}_2', \vec{e}_3'\} \cap \mathbb{S}^{d-1}$ – sufficiently close to \vec{e}_1 – as follows:

$$R_{\vec{t}}: \mathbb{R}^d \ni v \mapsto \mathcal{E}_{\vec{s}, \vec{s}'} \left(R_{(v_1(\vec{t}), v_2(\vec{t}), v_3(\vec{t}))^\top \times \mathbb{I}_{d-3}} \right) \mathcal{E}_{\vec{s}, \vec{s}'}^{-1} v \in \mathbb{R}^d,$$

where the matrix $R_{(v_1(\vec{t}), v_2(\vec{t}), v_3(\vec{t}))^\top \times \mathbb{I}_{d-3}} \in \mathbb{R}^{d \times d}$ applies $R_{(v_1(\vec{t}), v_2(\vec{t}), v_3(\vec{t}))^\top}$ – as defined above for three dimensions – to the first three coordinates and leaves the other coordinates invariant. This matrix maps \vec{e}_1 to \vec{t} as desired and it is smooth in \vec{t} with the same Lipschitz constant as the matrix $R_{(v_1(\vec{t}), v_2(\vec{t}), v_3(\vec{t}))^\top}$. Therefore we may use this Lipschitz property to establish that

$$\|R_{\vec{s}} - R_{\vec{s}'}\| \lesssim \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}, \vec{s}')$$

as required. \square

Lemma A.5. For the matrix $U_{j,\ell} = R_{j,\ell}^{-1} D_{2^{-j}}$ we have the inverse estimate

$$|U_{j,\ell}^{-1} \vec{s}| \leq w(\lambda), \quad (\text{A.5})$$

where the $w(\lambda) = 1 + 2^j |\vec{s} \cdot \vec{s}_{j,\ell}|$ is again the weight of the preconditioning matrix.

Additionally, for $U_{j',\ell'} = R_{j',\ell'}^{-1} D_{2^{-j'}}$ with $R_{j',\ell'}$ such that (A.3) holds for $\vec{s} = \vec{s}_{j,\ell}$ and $\vec{s}' = \vec{s}_{j',\ell'}$, we have

$$\|U_{j,\ell}^{-1} U_{j',\ell'}\| \lesssim \max(2^{j-j'}, 1) + 2^j \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}), \quad (\text{A.6})$$

and

$$|U_{j,\ell}^{-1} \vec{s}| \lesssim \max(2^{j-j'}, 1) (w(\lambda') + 2^{j'} \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'})).$$

Proof. For (A.5), the components have to be computed individually (using the orthogonality of the rotation),

$$U_{j,\ell}^{-1} \vec{s} = D_{2^j} R_{j,\ell} \vec{s} = D_{2^j} \begin{pmatrix} \vec{e}_1 \cdot R_{j,\ell} \vec{s} \\ \vec{e}_2 \cdot R_{j,\ell} \vec{s} \\ \vdots \end{pmatrix} = D_{2^j} \begin{pmatrix} R_{j,\ell}^{-1} \vec{e}_1 \cdot \vec{s} \\ R_{j,\ell}^{-1} \vec{e}_2 \cdot \vec{s} \\ \vdots \end{pmatrix} = D_{2^j} \begin{pmatrix} \vec{s}_{j,\ell} \cdot \vec{s} \\ R_{j,\ell}^{-1} \vec{e}_2 \cdot \vec{s} \\ \vdots \end{pmatrix} = \begin{pmatrix} 2^j \vec{s}_{j,\ell} \cdot \vec{s} \\ R_{j,\ell}^{-1} \vec{e}_2 \cdot \vec{s} \\ \vdots \end{pmatrix},$$

and consequently, since all but the first component have modulus less than 1,

$$|U_{j,\ell}^{-1} \vec{s}| \leq \max(2^j |\vec{s} \cdot \vec{s}_{j,\ell}|, 1) \leq w(\lambda).$$

Denoting the identity by \mathbb{I} , we begin the proof of (A.6) by considering the matrix $R_{j,\ell} R_{j',\ell'}^{-1}$ – exploiting the orthogonality of the $R_{j,\ell}$ and Lemma A.4 to yield

$$\|R_{j,\ell} R_{j',\ell'}^{-1} - \mathbb{I}\| = \|R_{j',\ell'} - R_{j,\ell}\| \lesssim \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}).$$

Thus, we can estimate

$$\begin{aligned} \|U_{j,\ell}^{-1} U_{j',\ell'}\| &= \|D_{2^j} R_{j,\ell} R_{j',\ell'}^{-1} D_{2^{-j'}}\| = \|D_{2^{j-j'}} + D_{2^j} (R_{j,\ell} R_{j',\ell'}^{-1} - \mathbb{I}) D_{2^{-j'}}\| \\ &\lesssim \max(2^{j-j'}, 1) + 2^j \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}). \end{aligned}$$

Finally, for the last inequality, we compute

$$U_{j,\ell}^{-1} = D_{2^j} R_{j,\ell} = D_{2^{j-j'}} D_{2^{j'}} R_{j,\ell} R_{j',\ell'}^{-1} R_{j',\ell'} = D_{2^{j-j'}} (U_{j',\ell'}^{-1} + D_{2^{j'}} (R_{j,\ell} R_{j',\ell'}^{-1} - \mathbb{I}) R_{j',\ell'}),$$

and after multiplying with \vec{s} and taking the modulus, we use the above results to arrive at

$$|U_{j,\ell}^{-1} \vec{s}| \lesssim \max(2^{j-j'}, 1) (w(\lambda') + 2^{j'} \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'})),$$

which is what we wanted to prove. \square

Proposition A.6. For $j \geq 1$, the transformation $U_{j,\ell}^\top$ takes the “tiles” $P_{j,\ell}$ back into a bounded set around the origin (illustrated in Figure A.1),

$$U_{j,\ell}^\top P_{j,\ell} \subseteq \left[\frac{1}{2} \cos(\alpha_j), 2 \right] \times \mathcal{P}_{(\text{span}\{\vec{e}_1\})^\perp} (B_{\mathbb{R}^d}(0, 4)) \subseteq B_{\mathbb{R}^d}(0, 5). \quad (\text{A.7})$$

The Minkowski sums $P_{j,\ell}^m := P_{j,\ell} + B_{\mathbb{R}^d}(0, 2^m)$ behave similarly,

$$U_{j,\ell}^\top (P_{j,\ell}^m) \subseteq B_{\mathbb{R}^d}(0, 5 + 2^m). \quad (\text{A.8})$$

More importantly, we can calculate the opening angle of the cone containing $P_{j,\ell}$ as follows,

$$\alpha_j^m = \alpha_j + \arcsin\left(\frac{2^m}{2^{j-1}}\right) \leq c_\omega 2^{m-j}, \quad (\text{A.9})$$

as long as $j \geq m + 1$, where $c_\omega \leq \pi + 2$.

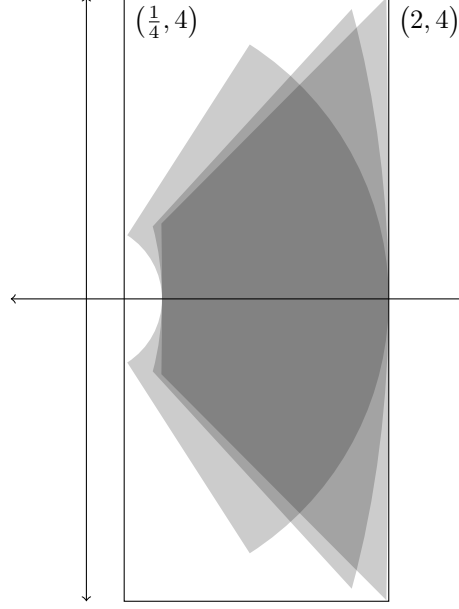


Figure A.1: $U_{j,\ell}^\top P_{j,\ell}$ for arbitrary ℓ and $j = 1, \dots, 3$. For better legibility, the y -axis is scaled down by a factor of 2.

Proof. From the definition of $\psi_{j,\ell}$ (see (4.2)), we see that its support $P_{j,\ell}$ is contained in the intersection between a spherical shell (between radii 2^{j-1} and 2^{j+1}) and a cone around $\vec{s}_{j,\ell}$ with opening angle $\alpha_j = 2^{-j+1}$. The rotation in $U_{j,\ell}^\top = D_{2^{-j}} R_{j,\ell}$ brings the axis of this cone into \vec{e}_1 . We see that the smallest value of η_1 for $\vec{\eta} \in U_{j,\ell}^\top P_{j,\ell}$ is $2^{-j} 2^{j-1} \cos(\alpha_j) = \frac{1}{2} \cos(\alpha_j) > \frac{1}{4}$ since $\alpha_j = 2^{-j+1} \leq 1 < \frac{\pi}{3}$ for $j \geq 1$.

The largest extent perpendicular to \vec{e}_1 can be calculated as

$$2^{j+1} \cos \alpha_j \sin \alpha_j = 2^j \sin 2\alpha_j \leq 2^j \cdot 2\alpha_j = 4,$$

which proves (A.7). (A.8) follows immediately because the contraction $D_{2^{-j}}$ can not enlarge the distance 2^m to $P_{j,\ell}$. We note that choosing a different

$$\tilde{R}_{j,\ell} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{R} \end{pmatrix} R_{j,\ell}$$

with $\tilde{R} \in \text{SO}(d-1)$ yields exactly the same set, since the rotation \tilde{R} leaves disks (in $d-1$ dimensions) invariant, i.e. $\tilde{R} B_{\mathbb{R}^{d-1}}(0, 4) = B_{\mathbb{R}^{d-1}}(0, 4)$.

By elementary geometric considerations (compare Figure A.2), we see that

$$\alpha_j^m = \alpha_j + \arcsin\left(\frac{2^m}{2^{j-1}}\right) \leq \alpha_j + \pi 2^{m-j} = 2^{m-j} (2^{-m+1} + \pi) \leq c_\omega 2^{m-j},$$

since $\arcsin x \leq \frac{\pi}{2} x$. The estimate can be made as long as $j \geq m+1$ and for all $m \geq 0$, $c_\omega \leq \pi + 2$. This finishes the proof. \square

Lemma A.7. *Let j', ℓ' as well as m, m' be fixed and denote $m_{>} := \max(m, m')$. If $j \geq m_{>} + 3$, the intersections $P_{j,\ell}^m \cap P_{j',\ell'}^{m'}$ can only be non-empty if j, ℓ satisfies*

$$|j - j'| \leq 2 \quad \text{and} \quad \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}) \leq 5c_\omega 2^{m_{>} - j}. \quad (\text{A.10})$$

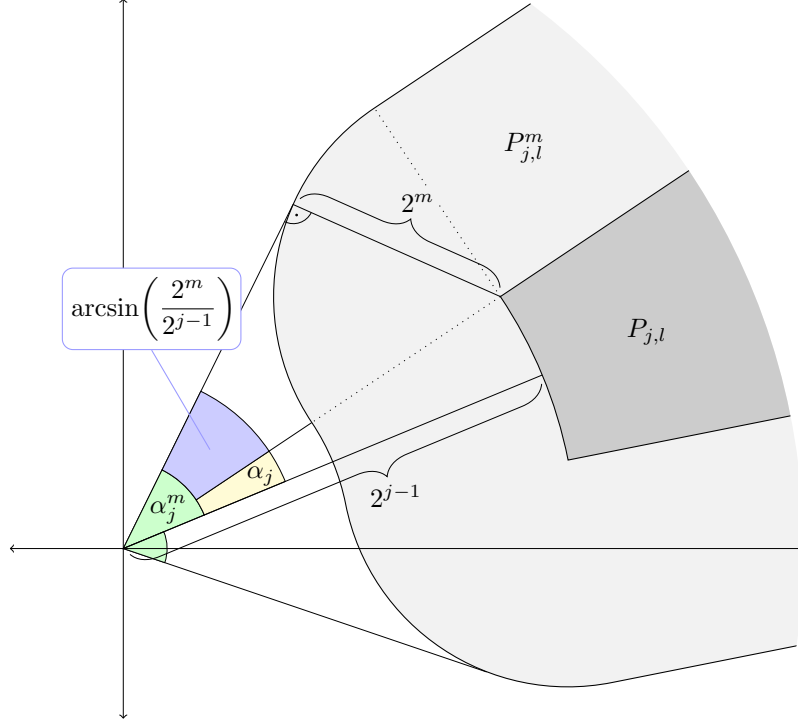


Figure A.2: The angle α_j^m can be computed explicitly

For the complementary case $j \leq m_{>} + 2$, we are not able to restrict the contributing indicies and have to assume the worst-case scenario. Put together, we have the inclusion

$$\{(j, \ell): P_{j,\ell}^m \cap P_{j',\ell'}^{m'} \neq \emptyset\} \subseteq \{(j, \ell): j \geq m_{>} + 3, \text{ (A.10) satisfied}\} \cup \{(j, \ell): j \leq m_{>} + 2, \ell \in \{0, \dots, L_j\}\}.$$

Proof. For fixed j', ℓ' , the a necessary condition for the intersection $P_{j,\ell}^m \cap P_{j',\ell'}^{m'}$ to be non-empty is

$$2^{j'+1} + 2^{m'} > 2^{j-1} - 2^m \quad \text{and} \quad 2^{j'-1} - 2^{m'} < 2^{j+1} + 2^m.$$

For $|j - j'| \leq 2$ this can always be satisfied for any $m \geq 0$. For $|j - j'| \geq 3$, one can check that it's only possible for $m_{>} > j - 3$. Said otherwise, if $j \geq m_{>} + 3$, then all intersections must satisfy $|j - j'| \leq 2$. This is illustrated in Figure A.3.

In terms of the angle, we observe that the Minkowski sums $P_{j,\ell}^m$ cannot anymore be easily described as the intersection of a spherical shell with a cone having its apex in the origin. However, it is still possible to find such a cone which contains $P_{j,\ell}^m$, having an enlarged opening angle $\alpha_j^m > \alpha_j$. By construction, we have that $\text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}) \leq \alpha_j^m + \alpha_{j'}^{m'}$ must be satisfied for the intersection to be non-empty. Naturally, these quantities can be estimated (see Proposition A.6),

$$\alpha_j^m \stackrel{\text{(A.9)}}{\leq} c_\omega 2^{m-j}, \quad \alpha_{j'}^{m'} \stackrel{\text{(A.9)}}{\leq} c_\omega 2^{m'-j'},$$

as long as $j \geq m + 1$ and $j' \geq m' + 1$, respectively. We can relate both quantities to j' , since by the above condition for j , we see that $\alpha_j^m + \alpha_{j'}^{m'} \leq (4 + 1)c_\omega 2^{m_{>} - j'}$.

Since j' is arbitrary, we cannot make any restrictions on it – however, we can use the fact that for $j \geq m_{>} + 3$, the above consideration in terms of scale still hold, and that in this case $|j - j'| \leq 2$ has to be satisfied. This gives us the desired condition $j' \geq m_{>} + 1$, which allows us to use the estimates for the

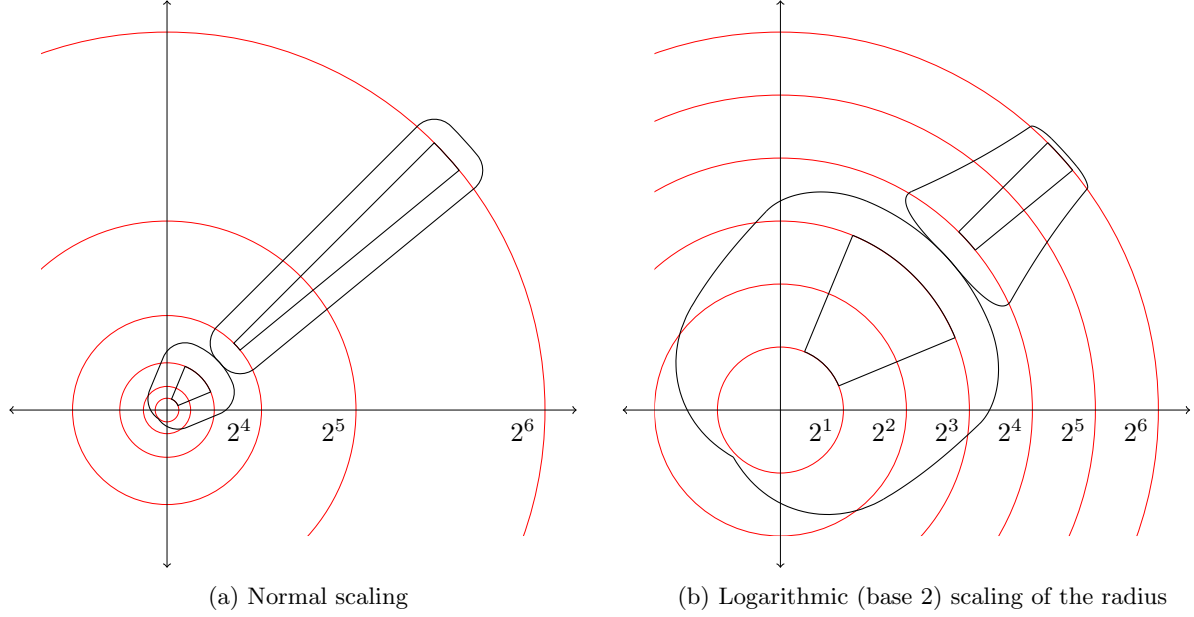


Figure A.3: Both subplots illustrate the argument of Lemma A.7, that for $|j - j'| = 3$, $m = m' = m_> = j_< = j_> - 3$ does not lead to an intersection

opening angles of the bounding cones. Collectively, these observations yield that $P_{j,\ell}^m \cap P_{j',\ell'}^{m'} \neq \emptyset$ implies

$$|j - j'| \leq 2 \quad \text{and} \quad \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}) \leq 5c_\omega 2^{m_>} \alpha_{j'},$$

as long as $j \geq m_> + 3$. In other words,

$$\{(j, \ell) : P_{j,\ell}^m \cap P_{j',\ell'}^{m'} \neq \emptyset\} \subseteq \{(j, \ell) : j \geq m_> + 3, \text{ (A.10) satisfied}\} \cup \{(j, \ell) : j \leq m_> + 2, \ell \in \{0, \dots, L_j\}\},$$

where we have assumed the worst-case scenario for $j < m_> + 3$. □

Appendix B A Suitable Choice of Window Functions

To prove that Assumption 4.7 is satisfiable, we show a possible way of constructing the window functions such that the assumption holds. Independent of the specific form of the function, the key property we need to show the desired estimates, is that the function $G(x)$ below is constant for $x < 1$.

Lemma B.1. *Choose*

$$t(x) := \frac{\exp\left(\frac{-1}{x^2}\right)}{\exp\left(\frac{-1}{x^2}\right) + \exp\left(\frac{-1}{(1-x)^2}\right)}.$$

as a C^∞ transition $t : [0, 1] \rightarrow [0, 1]$ with $t(0) = 0$ and $t(1) = 1$. Using this function, we construct

$$T(x) = \begin{cases} 1, & 0 \leq x < 1, \\ t(1-x), & 1 \leq x \leq 2, \\ 0, & 2 < x. \end{cases}$$

Setting

$$V^{(j,\ell)}(\vec{\xi}) := T\left(2^j \arccos\left(\frac{\vec{\xi}}{|\vec{\xi}|} \cdot \vec{s}_{j,\ell}\right)\right) = T(2^j \text{dist}_{\mathbb{S}^{d-1}}(\vec{\xi}, \vec{s}_{j,\ell})).$$

and

$$W(x) := \begin{cases} \sin(\frac{\pi}{2}t(x-1)) & 1 \leq x \leq 2, \\ \cos(\frac{\pi}{2}t(\frac{x}{2}-1)) & 2 < x \leq 4. \end{cases}$$

Then Assumption 4.7 holds.

Proof. Since the argument of the arctan is independent of the length, $V^{(j,\ell)}$ is homogeneous of degree zero – multiplicative constants in the argument do not change the result. Consequently, we may omit the normalising factor for this particular choice of $V^{(j,\ell)}$.

To estimate the derivatives of

$$\hat{\psi}_{(j,\ell)}(\vec{\eta}) := \hat{\psi}_{j,\ell}(U_{j,\ell}^{-\top} \vec{\eta}) = \frac{W(2^{-j}|D_{2^j}\vec{\eta}|) V^{(j,\ell)}(D_{2^j}\vec{\eta})}{\sqrt{\Phi(U_{j,\ell}^{-\top} \vec{\eta})}},$$

with arbitrary $R_{j,\ell}$ in $U_{j,\ell} = R_{j,\ell}^{-1} D_{2^{-j}}$, we have to estimate the derivatives of W , $V^{(j,\ell)}$ and Φ – all three share the restriction that $\vec{\eta}$ must lie in the support of the numerator of $\hat{\psi}_{j,\ell}$, $U_{j,\ell}^\top P_{j,\ell}$, see Definition 4.1.

For Φ , we see by (4.6), that for $\vec{\eta} \in U_{j,\ell}^\top P_{j,\ell}$, the sum only consists of only a few terms with indices “close to” j and ℓ ,

$$\Phi(U_{j,\ell}^{-\top} \vec{\eta}) = \sum_{\substack{j' \in \mathbb{N}_0: \\ |j-j'| \leq 1}} \sum_{\substack{\ell' \in \{0, \dots, L_{j'}\}: \\ \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'}) \leq 3\alpha_j}} W(2^{-j'} |D_{2^{j'}} \vec{\eta}|)^2 V^{(j',\ell')}(R_{j,\ell}^{-1} D_{2^j} \vec{\eta})^2.$$

Again, we can rewrite the function $V^{(j',\ell')}$,

$$\begin{aligned} V^{(j',\ell')}(R_{j,\ell}^{-1} D_{2^j} \vec{\eta}) &= T\left(2^{j'} \arccos\left(R_{j,\ell}^{-1} \frac{D_{2^j} \vec{\eta}}{|D_{2^j} \vec{\eta}|} \cdot \vec{s}_{j',\ell'}\right)\right) = T\left(2^{j'} \arccos\left(R_{j,\ell}^{-1} \frac{D_{2^j} \vec{\eta}}{|D_{2^j} \vec{\eta}|} \cdot R_{j',\ell'}^{-1} \vec{e}_1\right)\right) \\ &= T\left(2^{j'} \arccos\left(R_{j',\ell'} R_{j,\ell}^{-1} \frac{D_{2^j} \vec{\eta}}{|D_{2^j} \vec{\eta}|} \cdot \vec{e}_1\right)\right) = V^{(j'',0)}(R_{j',\ell'} R_{j,\ell}^{-1} D_{2^j} \vec{\eta}), \end{aligned}$$

where $R_{j',\ell'}$ can be any rotation taking $\vec{s}_{j',\ell'}$ to \vec{e}_1 – we choose it such that the transformation $R_{j',\ell'} R_{j,\ell}^{-1}$ is “close to” the identity, see Lemma A.4,

$$\|R_{j',\ell'} R_{j,\ell}^{-1} - \mathbb{I}\| \lesssim \text{dist}_{\mathbb{S}^{d-1}}(\vec{s}_{j,\ell}, \vec{s}_{j',\ell'})$$

Therefore, instead of proving the estimates for $V^{(j,0)}(D_{2^j} \vec{\eta})$ and $V^{(j',0)}(R_{j',\ell'} R_{j,\ell}^{-1} D_{2^j} \vec{\eta})$ separately, we consider $V^{(j',0)}(\vec{\zeta})$, where we let $M = (m_{i,k})_{i,k=1}^d$ be a general matrix and set $\vec{\zeta} := M D_{2^j} \vec{\eta}$ – in other words,

$$\zeta_i = m_{i,1} 2^j \eta_1 + m_{i,2} \eta_2 + \dots + m_{i,d} \eta_d, \quad i = 1, \dots, d$$

for the i -th entry. Alternatively, instead of applying $R_{j',\ell'} R_{j,\ell}^{-1}$ to $\frac{D_{2^j} \vec{\eta}}{|D_{2^j} \vec{\eta}|}$, we could have shifted both transformations to \vec{e}_1 , which would eliminate some difficulties (no chain rule necessary, see below), but complicate other estimates, especially (B.3).

As indicated above, the matrix M will be the identity or very close to it. In particular, for $M := R_{j',\ell'} R_{j,\ell}^{-1}$, the equivalence of all norms on $\mathbb{R}^{d \times d}$ implies that we can estimate the individual matrix entries,

$$|m_{i,k} - \delta_{i,k}| \leq \|R_{j',\ell'} R_{j,\ell}^{-1} - \mathbb{I}\|_{\text{Fro}} \leq c_{\text{Fro}} \|R_{j',\ell'} R_{j,\ell}^{-1} - \mathbb{I}\| \leq c_{\text{Fro}} 6c_{R_{\vec{s}}} 2^{-j} =: c_{\mathbb{I}} 2^{-j}, \quad (\text{B.1})$$

which allows us to choose j_0 such that for all $j \geq j_0$,

$$m_{i,i} \geq \frac{1}{2 \cos(\alpha_1)} = 0.925 \dots \quad \text{as well as} \quad 2^j \geq \sqrt{32(d-1)c_{\mathbb{I}}}. \quad (\text{B.2})$$

We can now investigate the support of (the derivatives of) $V^{(j',0)}(\vec{\zeta})$. The main point here is that since $T(x) = 1$ for $x < 1$, the derivatives vanish there as well and thus we obtain a lower bound for the angle,

$$\begin{aligned} \arccos\left(\frac{\zeta_1}{\sqrt{\zeta_1^2 + \dots + \zeta_d^2}}\right) \geq 2^{-j'} &\iff \zeta_1^2 \leq (\zeta_1^2 + \dots + \zeta_d^2) \cos(2^{-j'})^2 \\ &\iff \zeta_1^2 \sin(2^{-j'})^2 \leq (\zeta_2^2 + \dots + \zeta_d^2) \cos(2^{-j'})^2 \\ &\iff \zeta_1^2 \tan(2^{-j'})^2 \leq \zeta_2^2 + \zeta_3^2 + \dots + \zeta_d^2. \end{aligned} \quad (\text{B.3})$$

We want to derive a lower bound for the right hand side independently of j , and thus we return to

$$|\zeta_1| = |m_{1,1}2^j\eta_1 + m_{1,2}\eta_2 + \dots + m_{1,d}\eta_d| \geq m_{1,1}2^j|\eta_1| - |m_{1,2}\eta_2 + \dots + m_{1,d}\eta_d|,$$

where the second term can be estimated as follows

$$|m_{1,2}\eta_2 + \dots + m_{1,d}\eta_d| \stackrel{(\text{B.1})}{\leq} (d-1)c_{\mathbb{I}}2^{-j}\sqrt{\eta_2^2 + \dots + \eta_d^2} \stackrel{(\text{A.7})}{\leq} 2^{-j+2}(d-1)c_{\mathbb{I}}.$$

Together with (B.2) and (A.7), we see that

$$|\zeta_1| \geq 2^j \frac{\cos(\alpha_j)}{4 \cos(\alpha_1)} - 2^{-j+2}(d-1)c_{\mathbb{I}} \geq 2^j \left(\frac{1}{4} - \frac{1}{8}\right) = 2^{j-3},$$

because $\cos(\alpha_j) \geq \cos(\alpha_1)$, yielding the desired lower estimate for $\vec{\zeta}$,

$$\sqrt{\zeta_2^2 + \zeta_3^2 + \dots + \zeta_d^2} \geq \tan(2^{-j'})|\zeta_1| \geq 2^{-j'+j-3} \geq 2^{-4},$$

since $\tan(x) \geq x$.

Now we are fully equipped to tackle the derivatives of

$$V^{(j',\ell')}(R_{j,\ell}^{-1}D_{2j}\vec{\eta}) = V^{(j',0)}(T D_{2j}\vec{\eta}) = V^{(j',0)}(\vec{\zeta}) = T\left(2^{j'} \arccos\left(\frac{\zeta_1}{|\vec{\zeta}|}\right)\right),$$

where it suffices to control the derivatives of $2^{j'} \arccos(\frac{\zeta_1}{|\vec{\zeta}|})$, since $G \in \mathcal{C}^\infty$ and does not depend on j . We calculate for $2 \leq k \leq d$

$$\begin{aligned} \frac{\partial}{\partial \eta_k} \arccos\left(\frac{\zeta_1}{|\vec{\zeta}|}\right) &= \frac{-1}{\sqrt{1 - \frac{\zeta_1^2}{\zeta_1^2 + \dots + \zeta_d^2}}} \left(\frac{m_{1,k}(\zeta_1^2 + \dots + \zeta_d^2) - \zeta_1^2 m_{1,k}}{\sqrt{\zeta_1^2 + \dots + \zeta_d^2}^3} - \frac{2\zeta_2 m_{2,k} + \dots + 2\zeta_d m_{d,k}}{2\sqrt{\zeta_1^2 + \dots + \zeta_d^2}^3} \right) \\ &= \frac{-\sqrt{\zeta_1^2 + \dots + \zeta_d^2}}{\sqrt{\zeta_2^2 + \dots + \zeta_d^2}} \left(\frac{m_{1,k}(\zeta_2^2 + \dots + \zeta_d^2)}{\sqrt{\zeta_1^2 + \dots + \zeta_d^2}^3} - \frac{\zeta_2 m_{2,k} + \dots + \zeta_d m_{d,k}}{\sqrt{\zeta_1^2 + \dots + \zeta_d^2}^3} \right) \\ &= \frac{-m_{1,k}\sqrt{\zeta_2^2 + \dots + \zeta_d^2}}{\zeta_1^2 + \dots + \zeta_d^2} - \frac{1}{(\zeta_1^2 + \dots + \zeta_d^2)\sqrt{\zeta_2^2 + \dots + \zeta_d^2}} (\zeta_2 m_{2,k} + \dots + \zeta_d m_{d,k}). \end{aligned}$$

The case $k = 1$ is the same, except for an additional factor 2^j everywhere, due to the definition of $\vec{\zeta}$.

By induction, we extend this to higher derivatives, using standard multi-index notation. We only care about the kinds of terms that will appear, but not their respective weights – exact calculation is certainly possible, but only by not investigating the constants are we able to present a proof of acceptable length.

As might be expected from looking at the definition of $\vec{\zeta}$ (and the first derivative above), we will get a $m_{n,k}$ -factor for each derivative after η_k , depending on which component ζ_n with $n \in \{1, \dots, d\}$ is being derived. To compress this notation, we let \vec{a} be a vector in $\{1, \dots, d\}^{|\alpha|}$ (as it is necessary to choose one component for each derivative), and denote

$$m_{\vec{a}, \alpha} := \prod_{k=1}^d \prod_{n=\sum_{r=1}^{k-1} \alpha_r + 1}^{\sum_{r=1}^k \alpha_r} m_{a_n, k}.$$

Apart from this, operations of the multi-indices are to be interpreted componentwise. Lastly, since $|j - j'| \leq 1$, we can replace $2^{j'}$ with 2^j up to a constant. This leads to the promised result of the induction,

$$\begin{aligned} \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} 2^j \arccos\left(\frac{\zeta_1}{|\vec{\zeta}|}\right) &= \sum_{\substack{\beta + \gamma + \delta = \alpha \\ \delta \leq \beta + \gamma \\ |\beta| \geq 1}} \sum_{\substack{\vec{a}' \in \{1, \dots, d\}^{|\beta| + |\delta|} \\ \vec{b} \in \{2, \dots, d\}^{|\gamma|}}} c_{\alpha, \beta, \vec{a}', \gamma, \vec{b}', \delta} \frac{2^{j(\alpha_1 + 1)} m_{\vec{a}', \beta} m_{\vec{b}, \gamma} \vec{\zeta}^{\beta + \gamma - \delta}}{(\zeta_1^2 + \dots + \zeta_d^2)^{|\beta|} \sqrt{\zeta_2^2 + \dots + \zeta_d^2}^{2|\gamma| + 1}} + \dots \\ &\dots + \sum_{\substack{\beta + \gamma + \delta = \alpha \\ \delta \leq \beta + \gamma \\ |\beta + \gamma + \delta| = |\alpha| - 1}} \sum_{\substack{\vec{a}' \in \{1, \dots, d\}^{|\beta| + |\delta| + 1} \\ \vec{b} \in \{2, \dots, d\}^{|\gamma|}}} d_{\alpha, \beta, \vec{a}', \gamma, \vec{b}', \delta} \frac{2^{j(\alpha_1 + 1)} m_{\vec{a}', \beta} m_{\vec{b}, \gamma} \vec{\zeta}^{\beta + \gamma - \delta}}{(\zeta_1^2 + \dots + \zeta_d^2)^{|\beta| + 1} \sqrt{\zeta_2^2 + \dots + \zeta_d^2}^{2|\gamma| - 1}}. \end{aligned}$$

Note, that in the second sum, the constant $d_{\alpha, \dots}$ is zero unless the vector \vec{a}' contains an entry which is 1 – in fact, all changes in the second sum boil down to requiring that at least once, the component of $\vec{\zeta}$ being derived was ζ_1 . In the first sum there is a somewhat complementary condition, namely that $c_{\alpha, \dots}$ is zero unless at least one entry of \vec{a}' does *not* contain 1.

The reward for this rather unwieldy formula is that we are now able to prove that it can be bounded independently of j . The goal is to balance the powers of 2^j in numerator and denominator – the other factor in the denominator is unproblematic because we derived $\sqrt{\zeta_2^2 + \dots + \zeta_d^2} \geq 2^{-4}$ above.

Since we know $|\zeta_1| \gtrsim 2^j$, the exponent of 2^j in the denominator is $2|\beta|$ and $2|\beta| + 2$, respectively. In the numerator of the first sum, powers of 2^j may appear in $\vec{\zeta}^{\beta + \gamma - \delta}$, and thus the exponent is at worst

$$\alpha_1 + 1 + |\beta + \gamma - \delta| \leq |\alpha| + |\beta| + |\gamma| - |\delta| + 1 = 2|\beta| + 2|\gamma| + 1,$$

using the decomposition of α and the fact that the “absolute value” of a multi-index is linear. At this point we have to exploit the proximity of M to the identity matrix, which implies that off-diagonal elements satisfy $m_{i,k} \leq c_{\pm} 2^{-j}$ as derived above. Since the powers of 2^j can only appear when deriving by η_1 , and $m_{\vec{b}', \gamma} \sim 2^{-j|\beta'|}$ can never yield an index 1 in the first component (by the restriction on \vec{b}'), it follows that the term $2|\gamma|$ can be eliminated. Lastly, as we mentioned above, one component of \vec{a}' must also not be equal to 1, and thus we can eliminate the term “+1” as well, and have balanced the powers of 2^j in the first term.

For the second sum we proceed similarly, the worst exponent in the numerator is

$$\alpha_1 + 1 + |\beta + \gamma - \delta| \leq |\alpha| + |\beta| + |\gamma| - |\delta| + 1 = 2|\beta| + 2|\gamma| + 2,$$

by the same argument as above, taking the different decomposition of α into account. The $2|\gamma|$ -term is eliminated like before and this concludes the hardest part.

Wrapping everything up, we now see that $V^{(j', 0)}(M D_{2j} \vec{\eta})$ has bounded derivatives for $\eta \in U_{j, \ell}^\top$. The derivatives of the function $W(2^{-j'} |D_{2j} \vec{\eta}|)$ are much easier to handle, because $W \in \mathcal{C}^\infty$ is benign and the inner derivatives

$$\frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} 2^{-j'} |D_{2j} \vec{\eta}| = \sum_{\substack{\beta + \gamma = \alpha \\ \gamma \leq \beta}} \frac{2^{-j' + j\alpha_1} \vec{\eta}^{\beta - \gamma}}{\sqrt{2^{2j} \eta_1^2 + \eta_2^2 + \dots + \eta_d^2}^{2|\beta| - 1}}$$

are (more than) balanced in terms of powers of 2^j since $\eta_1 \geq \frac{1}{4}$ and $|j - j'| \leq 1$. Together, this implies that the derivatives of $\Phi(U_{j,\ell}^{-\top} \vec{\eta})$ are bounded independently of j for $\vec{\eta} \in U_{j,\ell}^\top P_{j,\ell}$. For the numerator of $\psi_{(j,\ell)}$, we insert $M = \mathbb{I}$ and $j' = j$ into the above equations, which finally proves Assumption 4.7 for the presented choice of window functions. \square

Appendix C Derivatives and Convolutions

In the proof of Theorem 5.4, we need to explicitly calculate terms of the form $\Delta^n(fg)$. Although we are not aware of any reference, the formula below is almost certainly known already. However, it seems to be sufficiently non-standard to justify exploring it in a little bit more detail.

Proposition C.1. *For two sufficiently smooth functions $f, g: \mathbb{R}^d \rightarrow \mathbb{C}$, the product rule for the Laplacian reads as follows,*

$$\Delta^n(fg) = \sum_{j+k_1+k_2=n} 2^j \binom{n}{j, k_1, k_2} \sum_{|\alpha|=j} \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (\Delta^{k_1} f) \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (\Delta^{k_2} g), \quad (\text{C.1})$$

where $\binom{n}{j, k_1, k_2} = \frac{n!}{j! k_1! k_2!}$ is the trinomial coefficient and the differential operator $\frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha}$ in standard multi-index notation operates on the different coordinates of $\vec{\eta}$,

$$\frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} = \frac{\partial^{\alpha_1 + \dots + \alpha_d}}{\partial \eta_1^{\alpha_1} \dots \partial \eta_d^{\alpha_d}}.$$

Remark C.2. If we interpret ∇ as an operator taking a tensor of order i to a tensor of order $i + 1$, then (C.1) can be written even more compactly,

$$\Delta^n(fg) = \sum_{j+k_1+k_2=n} 2^j \binom{n}{j, k_1, k_2} \left\langle \nabla^j (\Delta^{k_1} f), \nabla^j (\Delta^{k_2} g) \right\rangle_{\text{Fro}},$$

where $\langle A, B \rangle_{\text{Fro}}$ is the sum over all componentwise products of the two tensors – i.e. the tensor analogue of the Frobenius inner product for matrices (which induces the Frobenius norm). We note that it's possible to generalise this formula to a product $\prod_{i=1}^q f_i$ with $q \in \mathbb{N}$ as well.

As an unrelated observation, setting $d = 1$ and comparing coefficients with the standard product rule yields a curious relation between bi- and trinomial coefficients,

$$\binom{2n}{\ell} = \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} 2^{\ell-2k} \binom{n}{\ell-2k, k, n+k-\ell} = \sum_{k=\max(\ell-n, 0)}^{\lfloor \frac{\ell}{2} \rfloor} \frac{2^{\ell-2k} n!}{(\ell-2k)! k! (n+k-\ell)!},$$

and in particular,

$$(2n)! = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{n-2k} (n!)^3}{(n-2k)! (k!)^2}.$$

Proof. The proof is a simple induction – the case $n = 0$ is trivial. For $n \rightarrow n + 1$ we consider

$$\begin{aligned}
\Delta^{n+1}(fg) &= \Delta \left(\sum_{j+k_1+k_2=n} 2^j \binom{n}{j, k_1, k_2} \langle \nabla^j(\Delta^{k_1} f), \nabla^j(\Delta^{k_2} g) \rangle_{\text{Fro}} \right) \\
&= \sum_{j+k_1+k_2=n} 2^j \binom{n}{j, k_1, k_2} \left(\langle \nabla^j(\Delta^{k_1+1} f), \nabla^j(\Delta^{k_2} g) \rangle_{\text{Fro}} \dots + \right. \\
&\quad \left. \dots + 2 \langle \nabla^{j+1}(\Delta^{k_1} f), \nabla^{j+1}(\Delta^{k_2} g) \rangle_{\text{Fro}} + \langle \nabla^j(\Delta^{k_1} f), \nabla^j(\Delta^{k_2+1} g) \rangle_{\text{Fro}} \right) \\
&= \sum_{j+k_1+k_2=n+1} 2^j \left(\binom{n}{j, k_1-1, k_2} + \binom{n}{j-1, k_1, k_2} + \binom{n}{j, k_1, k_2-1} \right) \langle \nabla^j(\Delta^{k_1} f), \nabla^j(\Delta^{k_2} g) \rangle_{\text{Fro}},
\end{aligned}$$

where for the last equation, we performed an index shift for each of the three summands independently (in k_1, j, k_2 , respectively) and were able to extend the range of the indices because all additional terms have a trinomial coefficient of zero (either one entry is negative or the sum $j + k_1 + k_2$ is greater than n). At this point we need an analogous result to a well-known property of Pascal's triangle, namely

$$\binom{n}{j-1, k_1, k_2} + \binom{n}{j, k_1-1, k_2} + \binom{n}{j, k_1, k_2-1} = \frac{n!(j+k_1+k_2)}{j!k_1!k_2!} = \frac{(n+1)!}{j!k_1!k_2!} = \binom{n+1}{j, k_1, k_2},$$

since $j + k_1 + k_2 = n + 1$. This finishes the proof. \square

An immediate corollary to Proposition C.1 is the following.

Corollary C.3. *Under the same assumptions as in Proposition C.1, we have*

$$|[\Delta^n(fg)](\vec{\eta})| \leq (4d)^n |f(\vec{\eta})|_{\mathcal{C}^{2n}} |g(\vec{\eta})|_{\mathcal{C}^{2n}} \leq (4d)^n \|f\|_{\mathcal{C}^{2n}} \|g\|_{\mathcal{C}^{2n}},$$

where $|f(\vec{\eta})|_{\mathcal{C}^{2n}} = \max_{0 \leq r \leq 2n} |f^{(r)}(\vec{\eta})|$ is the maximum of all derivatives up to order $2n$ of f at $\vec{\eta}$.

Proof. The sum $\sum_{|\alpha|=j}$ consists of d^j terms. This can be seen since the sum can also be interpreted as selecting j (possibly redundant) coordinates from $\{1, \dots, d\}$ – a vector in $\{1, \dots, d\}^j$. Alternatively, one can use multinomials for selecting multiplicities $\alpha_1, \dots, \alpha_d$ which sum to j .

Similarly, the operator Δ^k consists of $\binom{d+k-1}{k} \leq d^k$ terms (which corresponds to choosing k out of d elements with repetitions). With this in mind, taking the absolute value of (C.1) leads to

$$\begin{aligned}
|\Delta^n(f(\vec{\eta})g(\vec{\eta}))| &= \left| \sum_{j+k_1+k_2=n} 2^j \binom{n}{j, k_1, k_2} \sum_{|\alpha|=j} \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (\Delta^{k_1} f(\vec{\eta})) \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (\Delta^{k_2} g(\vec{\eta})) \right| \\
&\leq \sum_{j+k_1+k_2=n} 2^j \binom{n}{j, k_1, k_2} d^{j+k_1+k_2} |f(\vec{\eta})|_{\mathcal{C}^{j+2k_1}} |g(\vec{\eta})|_{\mathcal{C}^{j+2k_2}} \\
&\leq (4d)^n |f(\vec{\eta})|_{\mathcal{C}^{2n}} |g(\vec{\eta})|_{\mathcal{C}^{2n}} \leq (4d)^n \|f\|_{\mathcal{C}^{2n}} \|g\|_{\mathcal{C}^{2n}},
\end{aligned}$$

where we used the identity $\sum_{j+k_1+k_2=n} (2d)^j d^{k_1} d^{k_2} \binom{n}{j, k_1, k_2} = (4d)^n$, which is immediate by setting (a, b, c) to $(2d, d, d)$ in the trinomial expansion

$$(a + b + c)^n = \sum_{j+k_1+k_2=n} \binom{n}{j, k_1, k_2} a^j b^{k_1} c^{k_2}.$$

\square

Finally, we need the following auxiliary result for differentiating the pullbacks of a convolution.

Lemma C.4. *For any invertible linear transformation $U : \mathbb{R}^d \rightarrow \mathbb{R}^d$, the derivatives of the pullback of the convolution can be estimated as follows,*

$$\left| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} ((f * g)(U\vec{\eta})) \right| \leq \left\| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (f(U\cdot)) \right\|_\infty (\mathbb{1}_{\text{supp } f}(\cdot) * |g(\cdot)|)(U\eta).$$

Proof. We begin by computing

$$(f * g)(U\vec{\eta}) = \int f(\vec{\zeta}) g(U\vec{\eta} - \vec{\zeta}) d\vec{\zeta} = |\det U| \int f(U\vec{\xi}) g(U(\vec{\eta} - \vec{\xi})) d\vec{\xi} = |\det U| (f(U\cdot) * g(U\cdot))(\vec{\eta}).$$

We apply all derivatives of the convolution to the function f , thus

$$\frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} ((f * g)(U\vec{\eta})) = |\det U| \left(\frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (f(U\cdot)) * g(U\cdot) \right)(\vec{\eta}) = |\det U| \int \left(\frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} f(U\cdot) \right)(\vec{\xi}) g(U(\vec{\eta} - \vec{\xi})) d\vec{\xi}.$$

Estimating the derivatives of f by its maximal value times its support, we arrive at

$$\begin{aligned} \left| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} ((f * g)(U\vec{\eta})) \right| &\leq |\det U| \int \left\| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (f(U\cdot)) \right\|_\infty \mathbb{1}_{\text{supp } f}(U\vec{\xi}) |g(U(\vec{\eta} - \vec{\xi}))| d\vec{\xi} \\ &= \left\| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (f(U\cdot)) \right\|_\infty \int \mathbb{1}_{\text{supp } f}(\vec{\zeta}) |g(U\vec{\eta} - \vec{\zeta})| d\vec{\zeta} \\ &= \left\| \frac{\partial^{|\alpha|}}{\partial \vec{\eta}^\alpha} (f(U\cdot)) \right\|_\infty (\mathbb{1}_{\text{supp } f}(\cdot) * |g(\cdot)|)(U\eta), \end{aligned}$$

which finishes the proof. \square

References

- [BN07] L. Borup and M. Nielsen. Frame decomposition of decomposition spaces. *J. Fourier Anal. Appl.*, 13(1):39–70, 2007.
- [Can98] E. Candès. Ridgelets: Theory and applications. PhD thesis, Stanford University, 1998.
- [Can01] E. Candès. Ridgelets and the representation of mutilated Sobolev functions. *SIAM J. Math. Anal.*, 33(2):347–368, 2001.
- [CD05a] E. Candès and D.L. Donoho. Continuous curvelet transform: I. Resolution of the Wavefront Set. *Appl. Comput. Harmon. Anal.*, 19(2):198–222, 2005.
- [CD05b] E. Candès and D.L. Donoho. Continuous curvelet transform: II. Discretization and frames. *Appl. Comput. Harmon. Anal.*, 19(2):198–222, 2005.
- [CDD01] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods for elliptic operator equations: convergence rates. *Math. Comp.*, 70(233):27–75, 2001.
- [CDDY06] E. Candès, L. Demanet, D.L. Donoho, and L. Ying. Fast discrete curvelet transforms. *Mult. Model. Simul.*, 5:861–899, 2006.
- [Dau92] I. Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
- [DeV98] R. DeVore. Nonlinear approximation. *Acta Numerica*, pages 51–150, 1998.

- [DFR07] S. Dahlke, M. Fornasier, and T. Raasch. Adaptive frame methods for elliptic operator equations. *Advances in Computational Mathematics*, 27(1):27–63, 2007.
- [DV05] M.N. Do and M. Vetterli. The contourlet transform: an efficient directional multiresolution image representation. *IEEE Trans. Image Proc.*, 14:2091–2106, 2005.
- [EG04] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.
- [EGO14] S. Etter, P. Grohs, and A. Obermeier. FFRT – a fast finite fourier transform for radiative transport. *Submitted*, 2014. Preprint available as a SAM Report (2014), ETH Zürich, http://www.sam.math.ethz.ch/sam_reports/index.php?id=2014-11.
- [GK14] P. Grohs and G. Kutyniok. Parabolic molecules. *Foundations of Computational Mathematics*, 14(2):299–337, 2014.
- [GKKS14] P. Grohs, S. Keiper, G. Kutyniok, and M. Schäfer. α -molecules. *Submitted*, 2014. Preprint available as a SAM Report (2014), ETH Zürich, http://www.sam.math.ethz.ch/sam_reports/index.php?id=2014-16.
- [GO14] P. Grohs and A. Obermeier. On the approximation of functions with line singularities by ridgelets. *In preparation*, 2014.
- [Gro11] P. Grohs. Ridgelet-type frame decompositions for sobolev spaces related to linear transport. *J. Fourier Anal. Appl.*, 2011.
- [GS11] P. Grohs and Ch. Schwab. Sparse twisted tensor frame discretization of parametric transport operators. 2011. Preprint available as a SAM Report (2011), ETH Zürich, http://www.sam.math.ethz.ch/sam_reports/index.php?id=2011-41.
- [HS78] P.R. Halmos and V.S. Sunder. *Bounded integral operators on L^2 spaces*, volume 96 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin-New York, 1978.
- [KL12] G. Kutyniok and D. Labate. *Shearlets: Multiscale Analysis for Multivariate Data*, chapter Introduction to Shearlets, pages 1–38. Birkhäuser, 2012.
- [KLLW05] G. Kutyniok, D. Labate, W.-Q Lim, and G. Weiss. Sparse multidimensional representation using shearlets. *Wavelets XI(San Diego, CA), SPIE Proc.*, 5914:254–262, 2005.
- [Li11] S. Li. Concise formulas for the area and volume of a hyperspherical cap. *Asian J. Math. Stat.*, 4(1):66–70, 2011.
- [Mod13] M.F. Modest. *Radiative heat transfer*. Academic press, 2013.
- [Ste04] R. Stevenson. Adaptive solution of operator equations using wavelet frames. *SIAM Journal on Numerical Analysis*, pages 1074–1100, 2004.